Energy spectrum of isotropic magnetohydrodynamic turbulence in the Lagrangian renormalized approximation

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0 Abstract

Quantitative estimates of the inertial-subrange statistics of MHD turbulence are given by using the Lagrangian renormalized approximation (LRA).
The estimate of energy spectrum is verified by DNS of forced MHD turbulence.

Outline of the talk

1 Introduction (Statistical theory of turbulence)

1.1 Governing equations of turbulence

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Navier-Stokes equations (in real space)

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} &= -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

$\mathbf{u}(\mathbf{x}, t)$:velocity field, $p(\mathbf{x}, t)$: pressure field,
ν: viscosity, $\mathbf{f}(\mathbf{x}, t)$: force field.

Navier-Stokes equations (in wavevector space)

$$\left(\frac{\partial}{\partial t} + \nu k^2 \right) u^i_{\mathbf{k}} = \int d\mathbf{p} d\mathbf{q} \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}) M^{iab}_{\mathbf{k}} u^a_{\mathbf{p}} u^b_{\mathbf{q}} + f^i_{\mathbf{k}}$$

$$M^{iab}_{\mathbf{k}} = -\frac{i}{2} \left[k_a P^{ib}_{\mathbf{k}} + k_b P^{ia}_{\mathbf{k}} \right], \qquad P^{ab}_{\mathbf{k}} = \delta_{ij} - \frac{k_i k_j}{k^2}.$$

Symbolically,

$$\left(\frac{\partial}{\partial t} + \nu L \right) u = M u u + f$$

1.2 Turbulence as a dynamical System

Characteristics of turbulence as a dynamical system

- Large number of degrees of freedom
- Nonlinear (modes are strongly interacting)
- Non-equilibrium (forced and dissipative)

Statistical mechanics of thermal equilibrium states can not be applied to turbulence.

- The law of equipartition do not hold.
- Probability distribution of physical variables strongly deviates from Gaussian (Gibbs distribution).

cf. (for thermal equilibrium states)

Thermodynamics

The macroscopic state is completely characterized by the free energy,

$$F(T,V,N).$$

Statistical mechanics

Macroscopic variables are related to microscopic characteristics (Hamiltonian).

$$F(T,V,N) = -kT \log Z(T,V,N)$$

Statistical theory of turbulence ?

What are the set of variables that characterize the statistical state of turbulence?

- ϵ? (Kolmogorov Theory ?)
- Fluctuation of ϵ? (Multifractal models?)

How to relate statistical variables to Navier-Stokes equations?

- Lagrangian Closures?

2 Lagrangian renormalized approximation (LRA)

2.1 Closure problem

Symbolically,

$$\frac{du}{dt} = \lambda M uu + \nu u$$

$\lambda := 1$ is introduced for convenience.

$$\frac{d}{dt}\langle u\rangle = \lambda M\langle uu\rangle + \nu\langle u\rangle,$$
$$\frac{d}{dt}\langle uu\rangle = \lambda M\langle uuu\rangle + \nu\langle uu\rangle,$$
$$\cdots$$

Equations for statistical quantities do not close.

$M\langle uuu\rangle$ should be expressed in terms of known quantity.

2.2 Solvable cases

- Weak turbulence (Wave turbulence)

$$\frac{du}{dt} = \lambda M u u + iLu, \qquad \left(\frac{d\tilde{u}}{dt} = \lambda \tilde{M} \tilde{u}\tilde{u}, \quad \tilde{u}(t) := e^{-iLt} u(t) \right)$$

The linear term iLu is dominant and the primitive λ-expansion may be justified in estimating $\lambda M \langle uuu \rangle$.

- Randomly advected passive scalar (or vector) model

$$\frac{du}{dt} = \lambda M v u + \nu u. \qquad (v\text{: advecting velocity field with given statistics})$$

When the correlation time scale τ_v of v tends to 0, the leading order of the primitive λ-expansion of $\lambda M \langle vuu \rangle$ becomes exact.

(One can also obtain closed equations for higher moments.)

2.3 Closure for Navier-Stokes turbulence

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Various closures are proposed for NS turbulence, but their mathematical foundations are not well established.

- Quasi normal approximation

$$\lambda M\langle uuu\rangle = \lambda^2\mathcal{F}[Q(t,t)]$$

$Q(t,s) := \langle u(t)u(s)\rangle$ correlation function.

  - Inappropriate since the closed equation derives negative energy spectrum.

- Direct interaction approximation (DIA) (Kraichnan, JFM **5** 497(1959))

$$\lambda M\langle uuu\rangle = \lambda^2\mathcal{F}[Q(t,s), G(t,s)]$$

$G(t,s)$ response function.

  - Derives an incorrect energy spectrum $E(k) \sim k^{-3/2}$. This is due to the inclusion of the sweeping effect of large eddies.

2.4 Lagrangian closures

- Abridged Lagrangian history direct interaction approximation (ALHDIA) (Kraichnan, Phys. Fluids **8** 575 (1965))
- Lagrangian renormalized approximation (LRA) (Kaneda, JFM **107** 131 (1981))

Key ideas of LRA

1. Lagrangian representatives Q^L and G^L.

$$M\langle vvv\rangle = \mathcal{F}[Q^L, G^L].$$

   - Representatives are different between ALHDIA and LRA.

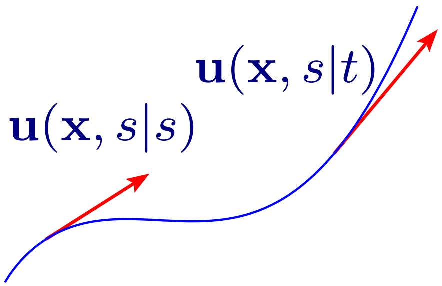
2. Mapping by the use of Lagrangian position function ψ.
3. Renormalized expansion.

Generalized Velocity

$\mathbf{u}(\mathbf{x}, s|t)$: velocity at time t of a fluid particle which passes $\mathbf{x}$ at time s.

s : labeling time

t : measuring time



Lagrangian Position function

$$\psi(\mathbf{y}, t; \mathbf{x}, s) = \delta^{(3)}(\mathbf{y} - \mathbf{z}(\mathbf{x}, s|t))$$

$\mathbf{z}(\mathbf{x}, s|t)$: position at time t of a fluid particle which passes $\mathbf{x}$ at time s.

$$\mathbf{u}(\mathbf{x}, s|t) = \int_{\mathcal{D}} d^3\mathbf{y}\ \mathbf{u}(\mathbf{y}, t)\psi(\mathbf{y}, t; \mathbf{x}, s)$$

2.6 Two-time two-point correlations

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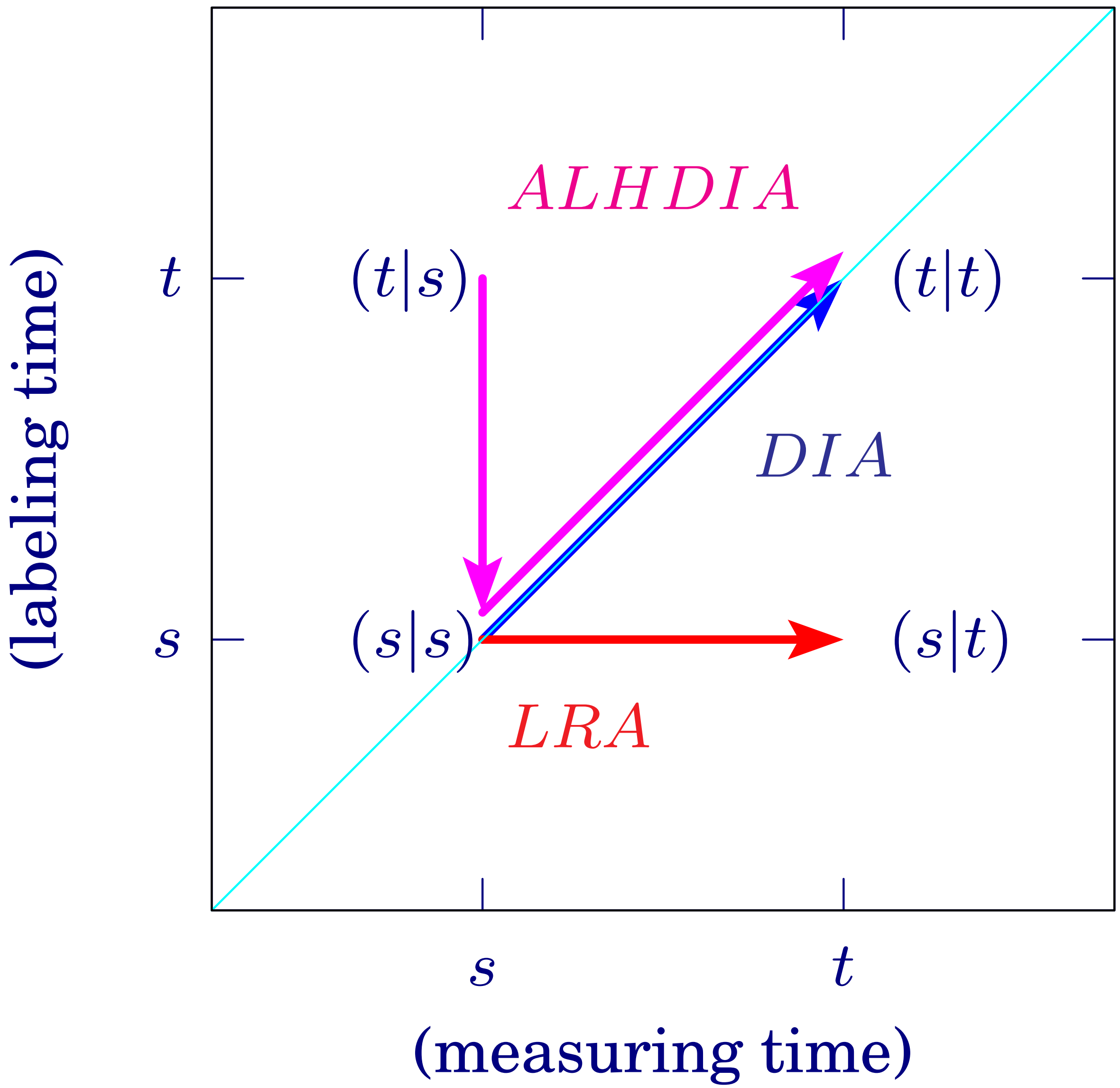

Representative Q (or Q^L)

$\langle \mathbf{u}(\mathbf{x}, t|t)\mathbf{u}(\mathbf{y}, s|s)\rangle$ (DIA)

$\langle \mathbf{u}(\mathbf{x}, t|t)\mathbf{u}(\mathbf{y}, t|s)\rangle$ (ALHDIA)

$\langle \mathcal{P}\mathbf{u}(\mathbf{x}, s|t)\mathbf{u}(\mathbf{y}, s|s)\rangle$ (LRA)

$\mathcal{P}\mathbf{u}$: solenoidal component of $\mathbf{u}$.

Similarly for G (or G^L).

2.7 Derivation of LRA

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(i) Primitive λ-expansion

$$\lambda M\langle uuu\rangle = \lambda^2\mathcal{F}^{(2)}[Q^{(0)},G^{(0)}] + \lambda^3\mathcal{F}^{(3)}[Q^{(0)},G^{(0)}] + O(\lambda^4),$$
$$\frac{\partial}{\partial t}Q^L(x,t;y,s) = \lambda^2\mathcal{I}^{(2)}[Q^{(0)},G^{(0)}] + \lambda^3\mathcal{I}^{(3)}[Q^{(0)},G^{(0)}] + O(\lambda^4),$$
$$\frac{\partial}{\partial t}G^L(x,t;y,s) = \lambda^2\mathcal{J}^{(2)}[Q^{(0)},G^{(0)}] + \lambda^3\mathcal{J}^{(3)}[Q^{(0)},G^{(0)}] + O(\lambda^4),$$

(ii) Inverse expansion

$$Q^{(0)} = Q^L + \lambda\mathcal{K}^{(1)}[Q^L,G^L] + O(\lambda^2), \qquad G^{(0)} = G^L + \lambda\mathcal{L}^{(1)}[Q^L,G^L] + O(\lambda^2)$$

(iii) Substitute (ii) into (i) (Renormalized expansion).

$$\lambda M\langle uuu\rangle = \lambda^2\mathcal{F}^{(2)}[Q^L,G^L] + O(\lambda^3),$$
$$\frac{\partial}{\partial t}Q^L(x,t;y,s) = \lambda^2\mathcal{I}^{(2)}[Q^L,G^L] + O(\lambda^3),$$
$$\frac{\partial}{\partial t}G^L(x,t;y,s) = \lambda^2\mathcal{J}^{(2)}[Q^L,G^L] + O(\lambda^3),$$

(iv) Truncate r.h.s.'s at the leading orders. (One may expect that $\lambda M\langle uuu\rangle$ depends on representatives gently when representatives are appropriately chosen.)

2.8 Consequences of LRA (1)

3D turbulence

- Kolmogorov energy spectrum

$$E(k) = K_o \epsilon^{2/3} k^{-5/3}, \qquad C_K \simeq 1.72.$$

(Kaneda, Phys. Fluids **29** 701 (1986))

2D turbulence

- Enstrophy cascade range

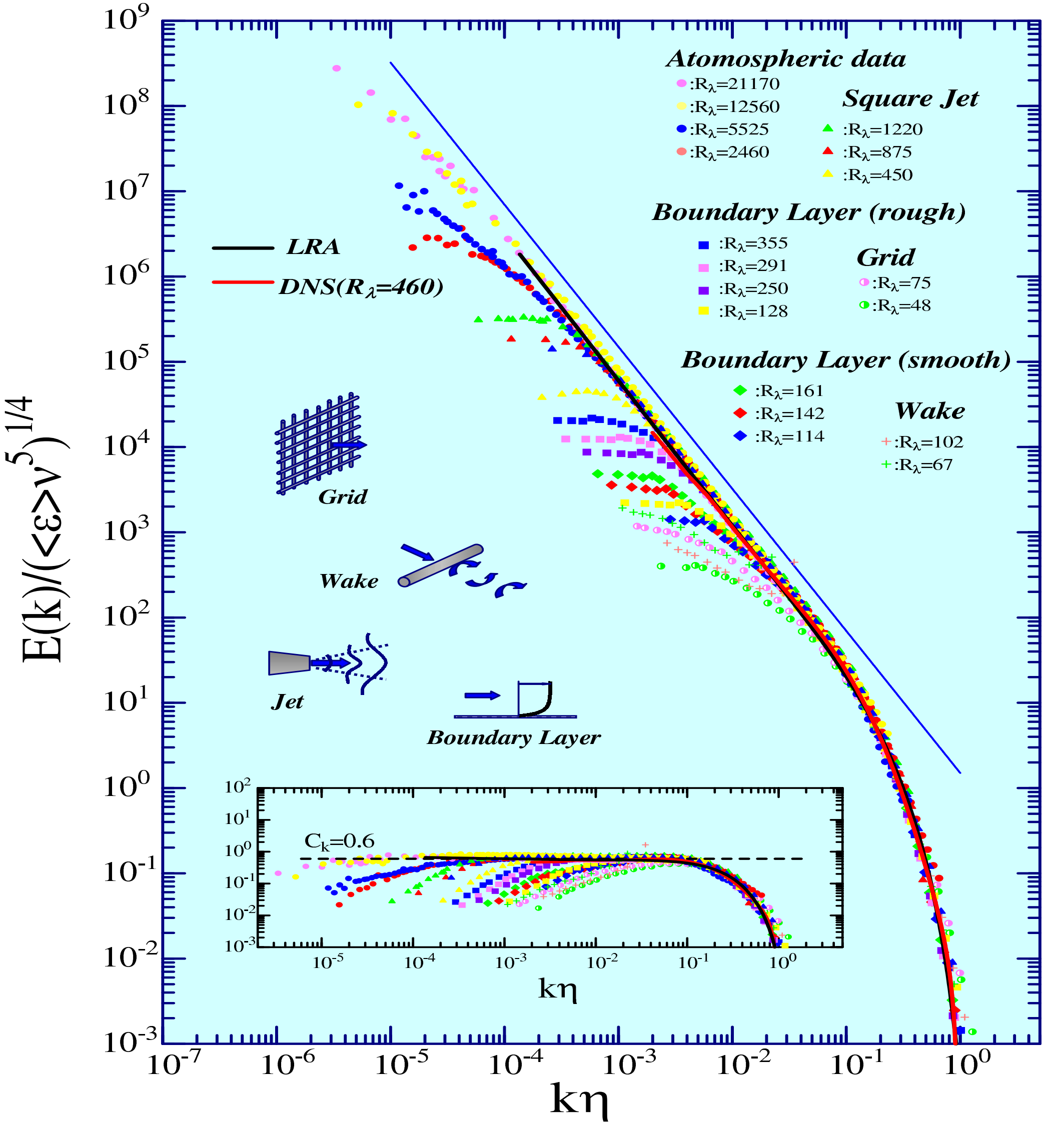
$$E(k) = \begin{cases} C_K \eta^{2/3} k^{-3} [\ln(k/k_1)]^{-1/3}, & C_K \simeq 1.81 \\ C_L k^{-3} & (C_L \text{ is not a universal constant}) \end{cases},$$

depending on the large-scale flow condition.

- Inverse energy cascade range

$$E(k) = C_E \epsilon^{2/3} k^{-5/3}, \qquad C_E \simeq 7.41.$$

(Kaneda, PF **30** 2672 (1987), Kaneda and Ishihara, PF **13** 1431 (2001))


Atomospheric data
:Rλ=21170
:Rλ=12560
:Rλ=5525
:Rλ=2460
Square Jet
:Rλ=1220
:Rλ=875
:Rλ=450
Boundary Layer (rough)
:Rλ=355
:Rλ=291
:Rλ=250
:Rλ=128
Grid
:Rλ=75
:Rλ=48
Boundary Layer (smooth)
:Rλ=161
:Rλ=142
:Rλ=114
Wake
:Rλ=102
:Rλ=67
LRA
DNS(Rλ=460)
Grid
Wake
Jet
Boundary Layer
Ck=0.6
kη
E(k)/(<ε>ν5)1/4
kη

Tsuji (2002)

2.9 Consequences of LRA (2)

LRA is also applied to

- Spectrum of passive scalar field advected by turbulence (3D / 2D) (Kaneda (1986), Kaneda (1987), Gotoh, J. Phys. Soc. Jpn. **58**, 2365 (1989)).
- Anisotropic modification of the velocity correlation spectrum due to homogeneous mean flow (Yoshida *et al.*, Phys. Fluids, **15**, 2385 (2003)).

Merits of LRA

- Fluctuation-dissipation relation $Q \propto G$ holds formally.
- The equations are simpler than ALHDIA.

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3 LRA for MHD

3.1 Magnetohydrodynamics (MHD)

- Interaction between a conducting fluid and a magnetic field.
- Geodynamo theory, solar phenomena, nuclear reactor, ...

Equations of incompressible MHD

$$
\begin{aligned}
\partial_t u_i + u_j \partial_j u_i &= B_j \partial_j B_i - \partial_i P + \nu_u \partial_j \partial_j u_i, \\
\partial_i u_i &= 0, \\
\partial_t B_i + u_j \partial_j B_i &= B_j \partial_j u_i + \nu_B \partial_j \partial_j B_i, \\
\partial_i B_i &= 0,
\end{aligned}
$$

$\mathbf{u}(\mathbf{x}, t)$: velocity field $\mathbf{B}(\mathbf{x}, t)$: magnetic field

ν_u : kinematic viscosity ν_B : magnetic diffusivity

3.2 Energy Spectrum: $k^{-3/2}$ or $k^{-5/3}$ or else?

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- Iroshnikov(1964) and Kraichnan(1965) derived IK spectrum

$$E^u(k) = E^B(k) = A\epsilon^{\frac{1}{2}} B_0^{\frac{1}{2}} k^{-\frac{3}{2}},$$

ϵ : total-energy dissipation rate, $\qquad B_0 = \sqrt{\frac{1}{3}\langle|\mathbf{B}|^2\rangle}$

based on a phenomenology which includes the effect of the Alfvén wave.

- Other phenomenologies (local anisotropy), including weak turbulence picture.
  (Goldreich and Sridhar (1994–1997), Galtier *et al.* (2000), etc.)

- Some results from direct numerical simulations (DNS) are in support of Kolmogorov-like $k^{-5/3}$-scaling.
  (Biskamp and Müller (2000), Müller and Grappin (2005))

3.3 Closure analysis for MHD turbulence

- Eddy-damped quasi-normal Markovian (EDQNM) approximation
  - Eddy-damping rate is so chosen to be consistent with the IK spectrum.
  - Incapable of quantitative estimate of nondimensional constant A.
  - Analysis of turbulence with magnetic helicity $\int_V d\mathbf{x}\mathbf{B}\cdot\mathbf{A}$ or cross helicity $\int_V d\mathbf{x}\mathbf{u}\cdot\mathbf{B}$.
    (Pouquet *et al.* (1976), Grappin *et al.* (1982,1983))
- LRA
  - A preliminary analysis suggests that LRA derives IK spectrum.
    (Kaneda and Gotoh (1987))
  - Present study
    * Quantitative analysis including the estimate of A.
    * Verification of the estimate by DNS.

3.4 Lagrangian variables

$$X_i^\alpha(\mathbf{x}, s|t) = \int_{\mathcal{D}} d^3\mathbf{x}' X_i^\alpha(\mathbf{x}', t)\psi(\mathbf{x}', t; \mathbf{x}, s), \qquad X_i^u := u_i, \qquad X_i^B := B_i,$$

Q: 2-point 2-time Lagrangian correlation function

G: Lagrangian response function

$$Q_{ij}^{\alpha\beta}(\mathbf{x}, t; \mathbf{x}', t') := \begin{cases} \langle [\mathcal{P}X^\alpha]_i(\mathbf{x}, t'|t) X_j^\beta(\mathbf{x}', t')\rangle & (t \geq t') \\ \langle X_i^\alpha(\mathbf{x}, t)[\mathcal{P}X^\beta]_j(\mathbf{x}', t|t')\rangle & (t < t') \end{cases},$$

$$\langle [\mathcal{P}\delta X^\alpha]_i(\mathbf{x}, t'|t)\rangle = G_{ij}^{\alpha\beta}(\mathbf{x}, t; \mathbf{x}', t')[\mathcal{P}\delta X^\beta]_j(\mathbf{x}', t'|t') \qquad (t \geq t'),$$

$\mathcal{P}$: Projection to the solenoidal part.

In Fourier Space

$$\hat{Q}_{ij}^{\alpha\beta}(\mathbf{k}, t, t') := (2\pi)^{-3} \int d^3(\mathbf{x} - \mathbf{x}') e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} Q_{ij}^{\alpha\beta}(\mathbf{x}, t, \mathbf{x}', t'),$$

$$\hat{G}_{ij}^{\alpha\beta}(\mathbf{k}, t, t') := \int d^3(\mathbf{x} - \mathbf{x}') e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} G_{ij}^{\alpha\beta}(\mathbf{x}, t, \mathbf{x}', t').$$

Isotropic turbulence without cross-helicity.

$$Q^{uu}_{ij}(\mathbf{k},t,s)=\frac{1}{2}Q^{u}(k,t,s)P_{ij}(\mathbf{k}),\qquad Q^{BB}_{ij}(\mathbf{k},t,s)=\frac{1}{2}Q^{B}(k,t,s)P_{ij}(\mathbf{k}),$$
$$Q^{uB}_{ij}(\mathbf{k},t,s)=Q^{Bu}_{ij}(\mathbf{k},t,s)=0$$
$$G^{uu}_{ij}(\mathbf{k},t,s)=G^{u}(k,t,s)P_{ij}(\mathbf{k}),\qquad G^{BB}_{ij}(\mathbf{k},t,s)=G^{B}(k,t,s)P_{ij}(\mathbf{k}),$$
$$G^{uB}_{ij}(\mathbf{k},t,s)=G^{Bu}_{ij}(\mathbf{k},t,s)=0.$$

LRA equations

$$\left[\partial_t+2\nu^\alpha k^2\right]Q^\alpha(k,t,t)=4\pi\iint_\triangle dp\,dq\frac{pq}{k}H^\alpha(k,p,q;t), \tag{1}$$
$$\left[\partial_t+\nu^\alpha k^2\right]Q^\alpha(k,t,s)=2\pi\iint_\triangle dp\,dq\frac{pq}{k}I^\alpha(k,p,q;t,s), \tag{2}$$
$$\left[\partial_t+\nu^\alpha k^2\right]G^\alpha(k,t,s)=2\pi\iint_\triangle dp\,dq\frac{pq}{k}J^\alpha(k,p,q;t,s), \tag{3}$$
$$G^\alpha(k,t,t)=1, \tag{4}$$

3.6 Response function

- Integrals in (2) and (3) diverge like $k_0^{3+a'}$ as $k_0 \to 0$.
  $Q^B(k) \propto k^{a'}$, k_0: the bottom wavenumber.

- No divergence due to $Q^u(k)$. (The sweeping effect of large eddies is removed.)

$$Q^u(k,t,s) = Q^B(k,t,s) = Q(k)g(kB_0(t-s)),$$
$$G^u(k,t,s) = G^B(k,t,s) = g(kB_0(t-s)),$$
$$g(x) = \frac{J_1(2x)}{x},$$

- Lagrangian correlation time $\tau(k)$ scales as $\tau(k) \sim (kB_0)^{-1}$.

3.7 Energy Spectrum in LRA

Energy spectrum

$$E^{\alpha}(k) = 2\pi k^2 Q^{\alpha}(k)$$

Energy Flux into wavenumbers $> k$

$$\begin{aligned}\Pi(k,t) &= \int_k^{\infty} dk' \, \frac{\partial}{\partial t}\bigg|_{NL} [E^u(k,t) + E^B(k,t)] \\ &= \int_k^{\infty} dk' \int_0^{\infty} dp' \int_{|p'-k'|}^{p'+k'} dq' \, T(k',p',q')\end{aligned}$$

Constant energy flux

$$\Pi(k,t) = \epsilon$$

$$\boxed{E^u(k) = E^B(k) = A\epsilon^{1/2} B_0^{1/2} k^{-3/2},}$$

The value of A is determined.

3.8 Energy flux and triad interactions

$$\epsilon = \Pi(k) = \int_0^\infty dp' \int_{|p'-k'|}^{p'+k'} dq'\, T(k',p',q')$$

$$\epsilon = \int_1^\infty \frac{d\alpha}{\alpha} W(\alpha) \qquad \alpha := \frac{\max(k',p',q')}{\min(k',p',q')}$$

- Triad interactions in MHD turbulence are slightly more local than those in HD turbulence.

3.9 Eddy viscosity and eddy magnetic diffusivity

$$H_{ij}^{\alpha\beta>}(\mathbf{k}, k_c, t) := \int_{\mathbf{p},\mathbf{q}}^{\triangle>} H_{ij}^{\alpha\beta}(\mathbf{k}, \mathbf{p}, \mathbf{q}, t),$$

$$\left(\partial_t Q_{i\,j}^{\alpha\beta}(\mathbf{k}, t, t) = \int_{\mathbf{p},\mathbf{q}}^{\triangle} \left[H_{i\,j}^{\alpha\beta}(\mathbf{k}, \mathbf{p}, \mathbf{q}, t) + H_{j\,i}^{\beta\alpha}(-\mathbf{k}, -\mathbf{p}, -\mathbf{q}, t) \right] \right)$$

$$H_{ij}^{\alpha\beta>}(\mathbf{k}, k_c, t) = -\nu^{\alpha\gamma}(k_c, t) k^2 Q_{ij}^{\gamma\beta}(\mathbf{k}, t), \qquad (k/k_c \to 0)$$

$$\nu^u(k, k_c, t) = -\frac{H_{ii}^{uu>}(\mathbf{k}, k_c, t)}{k^2 Q^u(k, t)}, \qquad \nu^B(k, k_c, t) = -\frac{H_{ii}^{BB>}(\mathbf{k}, k_c, t)}{k^2 Q^B(k, t)}, \qquad (0 < k/k_c < 1)$$

$$\nu^u(k, k_c) := \epsilon^{1/2} B_0^{-1/2} k_c^{-3/2} f^u\left(\frac{k}{k_c}\right),$$

$$\nu^B(k, k_c) := \epsilon^{1/2} B_0^{-1/2} k_c^{-3/2} f^B\left(\frac{k}{k_c}\right),$$

- Kinetic energy transfers more efficiently than magnetic energy.

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4 Verification by DNS

4.1 Forced DNS of MHD

- $(2\pi)^3$ periodic box domain (512^3 grid-points).
- $\nu^u = \nu^B = \nu$
- Random forcing for **u** and **B** at large scales.
  - E^u and E^B are injected at the same rate.
  - Correlation time of the random force $\sim$ large-eddy-turnover time.
- Magnetic Taylor-microscale Reynolds number: $R_\lambda^M := \sqrt{\frac{20 E^u E^B}{3\epsilon\nu}} = 188.$

4.2 Energy spectra in DNS

$$E(k) := E^u(k) + E^B(k), \qquad E^R(k) = E^u(k) - E^B(k).$$

- $E(k)$ is in good agreement with the LRA prediction,
- $E^R(k) \sim k^{-2}$. $E^u(k) \sim E^B(k)$ in small scales.

4.3 Comparison with other DNS

- Decaying DNS in Müller and Grappin (2005)
  - $E(k) \propto k^{-5/3}$ for $k > k_0$. $E^R(k_0)/E(k_0) \simeq 0.7$.
- Forced DNS in the present study
  - $E(k) \propto k^{-3/2}$ for $k > k_0$. $E^R(k_0)/E(k_0) \simeq 0.3$.

A 'higher' wavenumber regime is simulated in the present DNS.

5 Summary

Inertial-subrange statistics of MHD turbulence are analysis by using LRA.

- Lagrangian correlation time $\tau(k)$ scales as $\tau(k) \sim (kB_0)^{-1}$.
- Energy spectrum:

$$E^u(k,t) = E^B(k,t) = A\epsilon^{\frac{1}{2}} B_0^{\frac{1}{2}} k^{-\frac{3}{2}},$$

  - The value of A is estimated.
  - verified by forced DNS.
- Triad interactions are slightly more local than in HD turbulence.
- Eddy viscosity $>$ eddy magnetic diffusivity: