Workshop on New Perspectives in Quantum Turbulence:
experimental visualization and numerical simulation
Nagoya

Spectrum in Gross-Pitaevskii turbulence

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1 Quantum fluid (Introduction)

2 Closure Approximation

Quantum field equation

- Hamiltonian of interacting bosonic fields (^{4}He, Rb etc.) $\hat{\psi}(\boldsymbol{x}, t)$

$$\hat{H} = \int d\boldsymbol{x} \left[-\hat{\psi}^\dagger \frac{\hbar^2}{2m} \nabla^2 \hat{\psi} - \mu \hat{\psi}^\dagger \hat{\psi} + \frac{g}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \right]$$

μ : chemical potential, g: coupling constant

- Heisenberg equation

$$i\hbar \frac{\partial \hat{\psi}}{\partial t} = -\left(\frac{\hbar^2}{2m} \nabla^2 + \mu \right) \hat{\psi} + g \hat{\psi}^\dagger \hat{\psi} \hat{\psi}$$

$$\hat{\psi} = \psi + \hat{\psi}', \qquad \psi := \langle \hat{\psi} \rangle$$

- **Order parameter** $\psi(\boldsymbol{x}, t)$
  - $\psi \neq 0$ for temperature $T < T_c$.
  - The order parameter contains information of superfluid component or Bose-Einstein condensate.

Gross-Pitaevskii equation

- The order parameter $\psi(x)$ ($x := \{\boldsymbol{x}, t\}$) obeys **Gross-Pitaevskii (GP) equation**

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(x) = -\frac{\hbar^2}{2m}\nabla^2\psi(x) - \mu\psi(x) + g|\psi(x)|^2\psi(x).$$

- Transformation of variables

$$\psi(x) = \sqrt{n(x)}\ \mathrm{e}^{i\varphi(x)}, \quad \boldsymbol{v}(x) := \frac{\hbar}{m}\nabla\varphi(x)$$

- Equations of motion for **Quantum fluid**

$$\frac{\partial}{\partial t}n(x) = -\boldsymbol{\nabla}\cdot(n(x)\boldsymbol{v}(x)), \quad \frac{\partial}{\partial t}\boldsymbol{v}(x) = -\boldsymbol{v}(x)\cdot\boldsymbol{\nabla}\boldsymbol{v}(x) - \boldsymbol{\nabla}p_q(x),$$

$$p_q(x) := -\frac{\mu}{m} + \frac{gn(x)}{m} - \frac{\hbar^2}{2m^2}\frac{\nabla^2\sqrt{n(x)}}{\sqrt{n(x)}}.$$

Constants of motion

Number of particles $\bar{n}$ and Energy $\bar{E}$

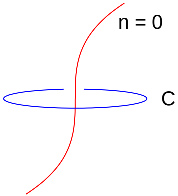
$$\begin{aligned}
\bar{n} &:= \frac{1}{V}\int \mathrm{d}\boldsymbol{x}|\psi(x)|^2, \\
\bar{E} &:= E_{\mathrm{K}}(t) + E_{\mathrm{I}}(t), \\
E_{\mathrm{K}}(t) &:= \frac{1}{V}\int \mathrm{d}\boldsymbol{x}\frac{\hbar^2}{2m}|\boldsymbol{\nabla}\psi(x)|^2, \\
E_{\mathrm{I}}(t) &:= \frac{1}{V}\int \mathrm{d}\boldsymbol{x}\ \frac{g}{2}|\boldsymbol{\psi}(x)|^4 = \frac{1}{V}\int \mathrm{d}\boldsymbol{x}\frac{g}{2}[n(x)]^2,
\end{aligned}$$

$E_{\mathrm{K}}(t)$: kinetic energy, $E_{\mathrm{I}}(t)$: interaction energy

Quantum fluid

Differences between quantum fluid and ordinary fluid obeying Navier-Stokes equation are

- No dissipation,
- Quasi-pressure term $p_q(x)$,
- No vorticity, $\boldsymbol{\omega}(x) := \boldsymbol{\nabla} \times \boldsymbol{v}(x) = 0$ where $n(x) \neq 0$,
- Vortex line for $n(x) = 0$ with a quantized circulation.



$$\oint_C \mathrm{d}\boldsymbol{l} \cdot \boldsymbol{v}(x) = \frac{2\pi\hbar}{m} k \qquad (k \in \mathbb{Z}).$$

Is the quantum fluid turbulence similar to the ordinary fluid turbulence?

Numerical simulation of GP equation

- Fourier transform of ψ

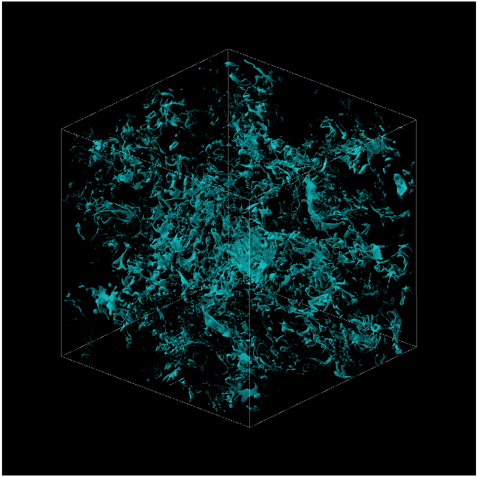
$$\psi_{\boldsymbol{k}}(t) := \int \mathrm{d}\boldsymbol{x} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \psi(x),$$

- GP equation with external force and dissipation in Fourier space representation.

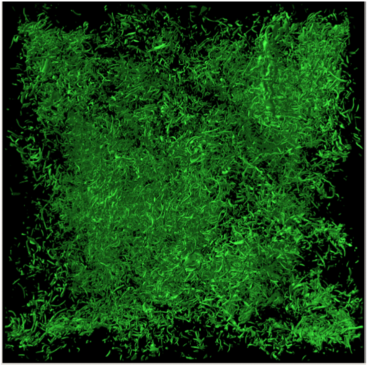
$$\frac{\partial}{\partial t}\psi_{\boldsymbol{k}} = -\mathrm{i}\xi^2 k^2 \psi_{\boldsymbol{k}} + \mathrm{i}\mu\psi_{\boldsymbol{k}} - \mathrm{i}g \int_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} \delta(\boldsymbol{k}+\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{r}) \psi^*_{\boldsymbol{p}} \psi_{\boldsymbol{q}} \psi_{\boldsymbol{r}} + D_{\boldsymbol{k}} + f_{\boldsymbol{k}}$$

$D_{\boldsymbol{k}}$: dissipation, $f_{\boldsymbol{k}}$: external force

Quantum and ordinary fluid turbulences



Low density region of a quantum fluid turbulence. Simulation with 512^3 grid points. (Yoshida and Arimitsu (2006))



cf. High vorticity region of a classical fluid turbulence. Simulation with 1024^3 grid points. (Kaneda and Ishihara (2006))

Spectra in Numerical Simulations

- Simulations with various kinds of $D_{\boldsymbol{k}}$ and $f_{\boldsymbol{k}}$.
- Spectrum of quantity X.

$$F^X(k) \propto \int_{\boldsymbol{k}'} \delta(k - |\boldsymbol{k}'|)\langle X(\boldsymbol{k}')X^*(\boldsymbol{k}')\rangle$$

  - Kobayashi and Tsubota (2005)
    - $F^{\mathrm{w}}(k) \sim k^{-5/3}$ ($\boldsymbol{w} = P[\sqrt{n}\boldsymbol{v}], P$ pjojection onto solenoidal component).
  - Yoshida and Arimitsu (2006)
    - $F^n(k) \sim k^{-3/2},\ F^{\psi}(k) \sim k^{-2/3}$.
  - Proment, Nazarenko and Onorato (2009)
    - $F^{\psi}(k) \sim k^{-1}$ or k^{-2}, depending on $D_{\boldsymbol{k}}$ and $f_{\boldsymbol{k}}$.
- **Scaling law of the Spectra in GP turbulence is unsettled.**

Theoretical approach

Doublet representation

$$\begin{pmatrix} \psi^+_{\boldsymbol{k}}(t) \\ \psi^-_{\boldsymbol{k}}(t) \end{pmatrix} := \mathrm{e}^{-L_{\boldsymbol{k}} t} \begin{pmatrix} \psi_{\boldsymbol{k}}(t) \\ \psi^*_{-\boldsymbol{k}}(t) \end{pmatrix}, \qquad L_{\boldsymbol{k}} := \mathrm{i}\left(-\frac{k^2}{2m} + \mu\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

GP equation in Fourier space

$$\frac{\partial}{\partial t}\psi^{\alpha}_{\boldsymbol{k}}(t) = g \int_{\boldsymbol{pqr}} \delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{r}} M^{\alpha\beta\gamma\zeta}_{\boldsymbol{kpqr}}(t)\psi^{\beta}_{\boldsymbol{p}}(t)\psi^{\gamma}_{\boldsymbol{q}}(t)\psi^{\zeta}_{\boldsymbol{r}}(t).$$

where $\int_{\boldsymbol{k}} := \int \mathrm{d}^3\boldsymbol{k}/(2\pi)^3$, $\delta_{\boldsymbol{k}} = (2\pi)^3\delta(\boldsymbol{k})$ and $\hbar = 1$.

$$M^{\alpha\beta\gamma\zeta}_{\boldsymbol{kpqr}}(t) := (\mathrm{e}^{-L_{\boldsymbol{k}}t})^{\alpha\alpha'}\tilde{M}^{\alpha'\beta'\gamma'\zeta'}_{\boldsymbol{kpqr}}(\mathrm{e}^{L_{\boldsymbol{p}}t})^{\beta'\beta}(\mathrm{e}^{L_{\boldsymbol{q}}t})^{\gamma'\gamma}(\mathrm{e}^{L_{\boldsymbol{r}}t})^{\zeta'\zeta},$$

$$\tilde{M}^{\alpha\beta\gamma\zeta}_{\boldsymbol{kpqr}} := \begin{cases} -\frac{\mathrm{i}}{3} & \text{for } (\alpha,\beta,\gamma,\zeta) \in \{(+,-,+,+),(+,+,-,+),(+,+,+,-)\} \\ \frac{\mathrm{i}}{3} & \text{for } (\alpha,\beta,\gamma,\zeta) \in \{(-,+,-,-),(-,-,+,-),(-,-,-,+)\} \\ 0 & \text{otherwise} \end{cases}.$$

Weak wave turbulence theory

- When $|\frac{\partial}{\partial t}\psi^{\pm}_{\boldsymbol{k}}| \ll |L_{\boldsymbol{k}}\psi^{\pm}_{\boldsymbol{k}}|$,

$$\psi^{\pm}_{\boldsymbol{k}}(t) \sim \text{const. in time}, \qquad \psi(x) \sim \int \mathrm{d}\boldsymbol{x}\psi^{+}_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}+L_{\boldsymbol{k}}t}.$$

- Correlation function

$$\langle \psi^{\alpha}_{\boldsymbol{k}}\psi^{\beta}_{-\boldsymbol{k}'}\rangle = Q^{\alpha\beta}_{\boldsymbol{k}}\delta_{\boldsymbol{k}-\boldsymbol{k}'},$$

- Spectrum

$$F(k) = \int_{\boldsymbol{k}'} \delta(k'-k)Q^{+-}_{\boldsymbol{k}'},$$

- **Weak wave turbulence (WWT) theory**
  - In the energy-transfer range,

$$F(k) \sim k^{-1}\left(\ln\frac{k}{k_{\mathrm{b}}}\right)^{-1/3}.$$

  - In the particle-number-transfer range,

$$F(k) \sim k^{-1/3}.$$

Strong turbulence

- GP turbulence
  - Weak wave turbulence (WWT) region: $|\frac{\partial}{\partial t}\psi_{\boldsymbol{k}}^{\pm}| \ll |L_{\boldsymbol{k}}\psi_{\boldsymbol{k}}^{\pm}|$,
  - Strong turbulence (ST) region: $|\frac{\partial}{\partial t}\psi_{\boldsymbol{k}}^{\pm}| \gg |L_{\boldsymbol{k}}\psi_{\boldsymbol{k}}^{\pm}|$.
- For the ordinary fluid turbulence, which is essentially strong turbulence, some spectral closure approximations are availiable.
  - $F^u(k) \propto k^{-5/3}$ in the energy-transfer range (Kolmogorov spectrum).

The aim of the present study is to derive the spectrum $F^{\psi}(k)$ of GP turbulence not only for the WWT region but for the **strong turbulence (ST) region** by means of a **spectral closure approximation**.

(K. Yoshida and T. Arimitsu, J. Phys. A: Math. Theor. **46** 335501 (2013))

1 Quantum fluid (Introduction)

2 Closure Approximation

Closure approximation

- Unclosed hierarchy of moments,

$$\frac{d}{dt}\langle\psi\rangle = gM\langle\psi\psi\psi\rangle, \qquad \frac{d}{dt}\langle\psi\psi\rangle = g\underline{M\langle\psi\psi\psi\psi\rangle}.$$

- Approximate $\underline{M\langle\psi\psi\psi\psi\rangle}$ as a function of lower order terms,

$$g\underline{M\langle\psi\psi\psi\psi\rangle} = g^2\mathcal{F}[Q(t,s), G(t,s)] + O(g^3)$$

  - Correlation function

$$\langle\psi^{\alpha}_{\boldsymbol{k}}(t)\psi^{\beta}_{-\boldsymbol{k}'}(t')\rangle = Q^{\alpha\beta}_{\boldsymbol{k}}(t,t')\delta_{\boldsymbol{k}-\boldsymbol{k}'},$$

  - Response function

$$\left\langle \frac{\delta\psi^{\alpha}_{\boldsymbol{k}}(t)}{\delta f^{\beta}_{\boldsymbol{k}'}(t')} \right\rangle = G^{\alpha\beta}_{\boldsymbol{k}}(t,t')\delta_{\boldsymbol{k}-\boldsymbol{k}'}.$$

  where $\delta f(t')$ is the infinitesimal disturbance added at time t'.

Invariance under global phase transformation

- For simplicity, let us assume that the statistical quantities are invariant under the global phase transformation,

$$\psi^{\alpha}_{\boldsymbol{k}}(t) \to \mathrm{e}^{\alpha \mathrm{i}\theta}\psi^{\alpha}_{\boldsymbol{k}}(t).$$

Then, by introducing $Q_{\boldsymbol{k}}(t,t')$ and $G_{\boldsymbol{k}}(t,t')$, we have

$$Q^{+-}_{\boldsymbol{k}}(t,t') = \mathrm{e}^{-2\mathrm{i}g\bar{n}(t-t')}Q_{\boldsymbol{k}}(t,t'), \quad Q^{-+}_{\boldsymbol{k}}(t,t') = \mathrm{e}^{2\mathrm{i}g\bar{n}(t-t')}Q^{*}_{-\boldsymbol{k}}(t,t'),$$
$$G^{++}_{\boldsymbol{k}}(t,t') = \mathrm{e}^{-2\mathrm{i}g\bar{n}(t-t')}G_{\boldsymbol{k}}(t,t'), \quad G^{--}_{\boldsymbol{k}}(t,t') = \mathrm{e}^{2\mathrm{i}g\bar{n}(t-t')}G^{*}_{-\boldsymbol{k}}(t,t'),$$

and otherwise 0.

Procedures for the closure approximation

(i) Expand Q and G in functional power series of the solutions $Q^{(0)}$ and $G^{(0)}$ for the zeroth-order in g.

$$Q = Q^{(0)} + \sum_{i=1}^{\infty} g^i Q^{(i)}(Q^{(0)}, G^{(0)}), \qquad G = G^{(0)} + \sum_{i=1}^{\infty} g^i G^{(i)}(Q^{(0)}, G^{(0)}),$$

$$\frac{\partial Q}{\partial t} = \sum_{i=0}^{\infty} g^i A^{(i)}(Q^{(0)}, G^{(0)}), \qquad \frac{\partial G}{\partial t} = \sum_{i=0}^{\infty} g^i B^{(i)}(Q^{(0)}, G^{(0)}).$$

(ii) Invert these expansions to obtain $Q^{(0)}$ and $G^{(0)}$ in functional power series of Q and G.

$$Q^{(0)} = Q + \sum_{i=1}^{\infty} g^i C^{(i)}(Q, G), \qquad G^{(0)} = G + \sum_{i=1}^{\infty} g^i D^{(i)}(Q, G).$$

(iii) Substitute these inverted expansions into the primitive expansions of $\mathrm{d}Q/\mathrm{d}t$ and $\mathrm{d}G/\mathrm{d}t$ to obtain the renormalized expansions.

$$\frac{\partial Q}{\partial t} = \sum_{i=0}^{\infty} g^i E^{(i)}(Q, G), \qquad \frac{\partial G}{\partial t} = \sum_{i=0}^{\infty} g^i F^{(i)}(Q, G).$$

(iv) Truncate these renormalized expansions at the lowest nontrivial order.

Closure equations (1)

$$
\begin{aligned}
&\frac{\partial}{\partial t} Q_{\boldsymbol{k}}(t,t') \\
=& g^2 \int_{-\infty}^{t} \mathrm{d}t'' \int_{\boldsymbol{pqr}} \delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{r}} \mathrm{e}^{\frac{\mathrm{i}}{2m}(k^2+p^2-q^2-r^2)(t-t'')} \\
&\times \Big[-2Q^*_{-\boldsymbol{p}}(t,t'')Q_{\boldsymbol{q}}(t,t'')G_{\boldsymbol{r}}(t,t'')Q_{\boldsymbol{k}}(t'',t') - 2Q^*_{-\boldsymbol{p}}(t,t'')G_{\boldsymbol{q}}(t,t'')Q_{\boldsymbol{r}}(t,t'')Q_{\boldsymbol{k}}(t'',t') \\
&\quad + 2G^*_{-\boldsymbol{p}}(t,t'')Q_{\boldsymbol{q}}(t,t'')Q_{\boldsymbol{r}}(t,t'')Q_{\boldsymbol{k}}(t'',t') + 2Q^*_{-\boldsymbol{p}}(t,t'')Q_{\boldsymbol{q}}(t,t'')Q_{\boldsymbol{r}}(t,t'')G^*_{\boldsymbol{k}}(t',t'') \Big],
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t} G_{\boldsymbol{k}}(t,t') \\
=& g^2 \int_{t'}^{t} \mathrm{d}t'' \int_{\boldsymbol{pqr}} \delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{r}} \mathrm{e}^{\frac{\mathrm{i}}{2m}(k^2+p^2-q^2-r^2)(t-t'')} \\
&\times \Big[-2Q^*_{-\boldsymbol{p}}(t,t'')Q_{\boldsymbol{q}}(t,t'')G_{\boldsymbol{r}}(t,t'')G_{\boldsymbol{k}}(t'',t') - 2Q^*_{-\boldsymbol{p}}(t,t'')G_{\boldsymbol{q}}(t,t'')Q_{\boldsymbol{r}}(t,t'')G_{\boldsymbol{k}}(t'',t') \\
&\quad + 2G^*_{-\boldsymbol{p}}(t,t'')Q_{\boldsymbol{q}}(t,t'')Q_{\boldsymbol{r}}(t,t'')G_{\boldsymbol{k}}(t'',t') \Big] + \delta(t-t'), \\
&G_{\boldsymbol{k}}(t,t') = 0 \qquad (t < t').
\end{aligned}
$$

Closure equations (2)

- Correlation function for the number density field,

$$\langle n_{\boldsymbol{k}}(t) n_{-\boldsymbol{k}}(t') \rangle - \langle n_{\boldsymbol{k}}(t) \rangle \langle n_{-\boldsymbol{k}}(t') \rangle = Q^n_{\boldsymbol{k}}(t, t') \delta_{\boldsymbol{k}-\boldsymbol{k}'},$$

$$\begin{aligned}
&\frac{\partial}{\partial t} Q^n_{\boldsymbol{k}}(t, t') \\
&= \mathrm{i} \int_{\boldsymbol{pq}} \delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} \frac{1}{2m}(p^2 - q^2) \mathrm{e}^{\frac{\mathrm{i}}{2m}(p^2-q^2)(t-t')} Q^*_{-\boldsymbol{p}}(t, t') Q_{\boldsymbol{q}}(t, t') \\
&+ g \int_{-\infty}^{t} \mathrm{d}t'' \int_{\boldsymbol{pqrs}} \delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} \delta_{\boldsymbol{k}-\boldsymbol{r}-\boldsymbol{s}} \frac{1}{m}(p^2 - q^2) \mathrm{e}^{\frac{\mathrm{i}}{2m}\left[(-p^2+q^2)(t-t'')+(r^2-s^2)(t'-t'')\right]} \\
&\times \Big[-G_{\boldsymbol{p}}(t, t'') Q^*_{-\boldsymbol{q}}(t, t'') Q^*_{\boldsymbol{r}}(t', t'') Q_{-\boldsymbol{s}}(t', t'') + Q_{\boldsymbol{p}}(t, t'') G^*_{-\boldsymbol{q}}(t, t'') Q^*_{\boldsymbol{r}}(t', t'') Q_{-\boldsymbol{s}}(t', t'') \\
&\quad + Q_{\boldsymbol{p}}(t, t'') Q^*_{-\boldsymbol{q}}(t, t'') G^*_{\boldsymbol{r}}(t', t'') Q_{-\boldsymbol{s}}(t', t'') - Q_{\boldsymbol{p}}(t, t'') Q^*_{-\boldsymbol{q}}(t, t'') Q^*_{\boldsymbol{r}}(t', t'') G_{-\boldsymbol{s}}(t', t'') \Big] \\
&+ O(g^2).
\end{aligned}$$

Time scale of the nonlinear term

- Time scales
  - $T_{\mathrm{L}}(k) := 2mk^{-2}$, time scale of the linear terms.
  - $T_{\mathrm{NL}}(k)$, time scale of $Q_{\boldsymbol{k}}(t,t')$ and $G_{\boldsymbol{k}}(t,t')$ with respect to $t-t'$.
- ST region: $T_{\mathrm{NL}}(k) \ll T_{\mathrm{L}}(k)$,
- Assume that the contribution from the low wavenumber region is dominant in the wavespace integration. Then,

$$\frac{\partial}{\partial t}Q_k(t,t') = g^2 \int_{-\infty}^{t} \mathrm{d}t'' \Big[n(t,t'')\Big]^2 \Big[-4G_k(t,t'')Q_k(t',t'') + 6Q_k(t,t'')G_k(t',t'')\Big],$$

$$\frac{\partial}{\partial t}G_k(t,t') = -4g^2 \int_{t'}^{t} \mathrm{d}t'' \Big[n(t,t'')\Big]^2 G_k(t,t'')G_k(t'',t') + \delta(t-t'),$$

where $n(t,t') = \int_{\boldsymbol{k}} Q_k(t,t')$.

- We have

$$T_{\mathrm{NL}}(k) = g^{-1}\bar{n}^{-1}$$

in ST region $k \ll k_*$, where

$$k_* := (2m)^{1/2} g^{1/2} \bar{n}^{1/2} \quad (T_{\mathrm{NL}}(k_*) = T_{\mathrm{L}}(k_*)).$$

Energy flux

- Energy flux (energy flowing into modes with wavenumber larger than K)

$$\Pi(K) := \frac{\partial}{\partial t} \int_{\substack{\boldsymbol{k} \\ k>K}} \left[\frac{k^2}{2m} Q_{\boldsymbol{k}}(t,t) + \frac{g}{2} Q^n_{\boldsymbol{k}}(t,t) \right].$$

- Symbolically,

$$\Pi(K) = g^2 \int_{\boldsymbol{kpqr},D} \delta_{\boldsymbol{k+p-q-r}} \int^t \mathrm{d}t' \mathrm{e}^{\frac{\mathrm{i}}{2m}(k^2+p^2-q^2-r^2)(t-t')} \frac{k^2}{m} \times Q_*(t,t')Q_*(t,t')Q_*(t,t')G_*(t,t')$$

$$(* = \boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \quad D : \text{a wavevector space region})$$

- When the contribution from the low wavenumber region is dominant,

$$\Pi(K) = g^2 \int_{\boldsymbol{kpq},D'} \delta_{\boldsymbol{k-p-q}} \int^t \mathrm{d}t' \mathrm{e}^{\frac{\mathrm{i}}{2m}(k^2 \pm p^2-q^2)(t-t')} \frac{k^2}{m} \times n(t,t')Q_*(t,t')Q_*(t,t')G_*(t,t')$$

- In the energy-transfer region,

$$\Pi(K) = \Pi \quad \text{(const.)}$$

Spectrum in the energy-transfer range

- ST region ($k \ll k_*$),

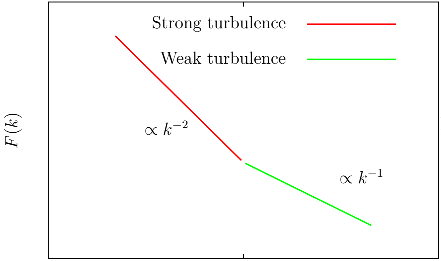
$$F(k) = C_1 (2m)^{1/2} g^{-1/2} |\Pi|^{1/2} k^{-2}.$$

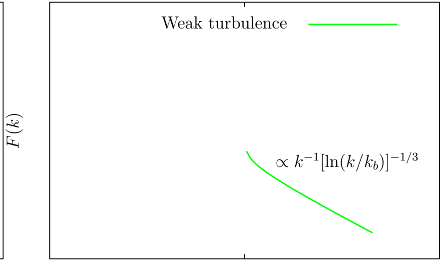
Probably, $\Pi > 0$.

- WWT region ($k \gg k_*$),

$$F(k) = \begin{cases} C_2 g^{-2/3} \Pi^{1/3} k^{-1} \left(\ln \frac{k}{k_\mathrm{b}} \right)^{-1/3} & \text{(low wavenumber marginal divergence)} \\ C_2' g^{-1} \bar{n}^{-1/2} \Pi^{1/2} k^{-1} & \text{(low wavenumber divergence)} \end{cases}$$

$\Pi > 0$.





Spectrum in the particle-number-transfer range

- Particle-number-flux (particles flowing into modes with wavenumber larger than K)

$$\Pi_{\rm n}(K) := \frac{\partial}{\partial t}\int_{\boldsymbol{k},k>K} Q_{\boldsymbol{k}}(t,t).$$

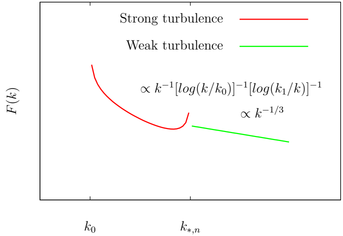
$$T_{{\rm NL,n}}(k) = g^{-1/2}|\Pi_{\rm n}|^{-1/2}, \qquad k_{*,{\rm n}} = (2m)^{1/2}g^{1/4}|\Pi_{\rm n}|^{1/4}.$$

- ST region ($k \ll k_{*,n}$)

$$F(k) = C_3 g^{-1/2}|\Pi_{\rm n}|^{1/2}k^{-1}[\ln(\frac{k}{k_0})]^{-1}[\ln(\frac{k_1}{k})]^{-1} \quad \text{(Probably } \Pi_{\rm n} > 0\text{)}.$$

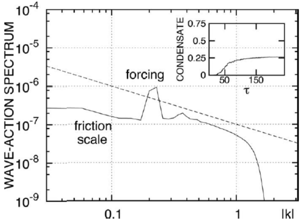
- WWT region ($k \gg k_{*,n}$)

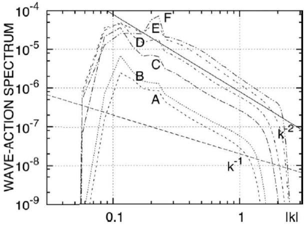
$$F(k) = C_4(2m)^{-1/3}g^{-2/3}|\Pi_{\rm n}|^{1/3}k^{-1/3} \quad (\Pi_{\rm n} < 0).$$



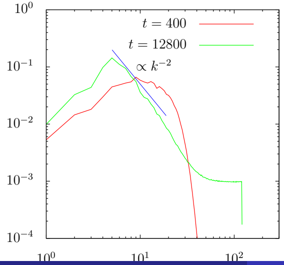
Numerical simulations

- Simulation with external forcing and dissipation [Proment, Nazarenko and Onorato (2009)]





- Simulation without external force and dissipation (Yoshida, in progress)



Summary and problems

Summary

- By means of a spectral closure, the spectra of GP turbulence are obtained for the ST/WWT regions in the energy-transfer/particle-number-transfer ranges.
- Some numerical simulations are in support of $F(k) \propto k^{-2}$ of the ST region in the energy-transfer range.

Problems

- Some correction to the spectrum of ST region in energy-transfer range is needed to cancel the energy flow from $E_{\rm I}$ to $E_{\rm K}$ and to maintain the statistical stationarity.
- Correction beyond the log correction is needed for ST region in particle-number-transfer range to eliminate the divergence of the integral.
- Since $\Pi_{\rm n} < 0$ for $k \gg k_{*,{\rm n}}$ and probably $\Pi_{\rm n} > 0$ for $k \ll k_{*,{\rm n}}$, their compatibility is questionable.