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"Computational Physics and New Perspectives in Turbulence"

Numerical Simulation of Quantum Fluid Turbulence

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0 Abstract

Gross-Pitaevskii (GP) equation describes the dynamics of low-temperature superfluids and Bose-Einstein Condensates (BEC). We performed a numerical simulation of turbulence obeying GP equation (Quantum fluid turbulence). Some results of the simulation are reported.

Outline

1 Background (Statistical Theory of Turbulence)

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Characteristics of (classical fluid) turbulence as a dynamical system are

- Large number of degrees of freedom
- Nonlinear (modes are strongly interacting)
- Non-equilibrium (forced and dissipative)

Why quantum fluid turbulence ?

- Another example of such a dynamical system. Another test ground for developing the statistical theory of turbulence.
  - What are in common and what are different between classical and quantum fluid turbulence?

2 Quantum fluid turbulence

2.1 Dynamics of order parameter

Hamiltonian of locally interacting boson field $\hat{\psi}(\mathbf{x}, t)$

$$\hat{H} = \int d\mathbf{x} \left[-\hat{\psi}^\dagger \frac{\hbar^2}{2m} \nabla^2 \hat{\psi} - \mu \hat{\psi}^\dagger \hat{\psi} + \frac{g}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \right]$$

μ : chemical potential, g: coupling constant

Heisenberg equation

$$i\hbar \frac{\partial \hat{\psi}}{\partial t} = - \left(\frac{\hbar^2}{2m} \nabla^2 + \mu \right) \hat{\psi} + g \hat{\psi}^\dagger \hat{\psi} \hat{\psi}$$

$$\hat{\psi} = \psi + \hat{\psi}', \qquad \psi = \langle \hat{\psi} \rangle$$

$\psi(\mathbf{x}, t)$: Order parameter

Gross-Pitaevskii (GP) equation

$$
\begin{aligned}
i\hbar\frac{\partial\psi}{\partial t} &= -\left(\frac{\hbar}{2m}\nabla^2+\mu\right)\psi+g|\psi|^2\psi, \\
\mu &= g\bar{n}, \qquad n=|\psi|^2
\end{aligned}
$$

$\bar{\cdot}$: volume average.

Normalization

$$
\tilde{\mathbf{x}}=\frac{\mathbf{x}}{L}, \qquad \tilde{t}=\frac{g\bar{n}}{\hbar}t, \qquad \tilde{\psi}=\frac{\psi}{\sqrt{\bar{n}}}
$$

Normalized GP equation

$$
i\frac{\partial\tilde{\psi}}{\partial\tilde{t}}=-\tilde{\xi}^2\tilde{\nabla}^2\tilde{\psi}-\tilde{\psi}+|\tilde{\psi}|^2\tilde{\psi}, \qquad \left(\xi=\frac{\hbar}{\sqrt{2mg\bar{n}}}, \qquad \tilde{\xi}=\frac{\xi}{L}\right)
$$

ξ: Healing length ($\sim 0.5\mathring{A}$ in Liquid ^{4}He)

Hereafter, $\tilde{\cdot}$ is omitted.

2.3 Quantum fluid velocity and quantized vortex line

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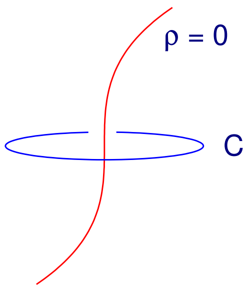
$$\psi(\mathbf{x},t) = \sqrt{\rho(\mathbf{x},t)}e^{i\varphi(\mathbf{x},t)}, \qquad \mathbf{v}(\mathbf{x},t) = 2\xi^2\nabla\varphi(\mathbf{x},t)$$

$$\begin{aligned} \frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\mathbf{v}) &= 0 \\ \frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} &= -\nabla p_q \qquad \left(p_q = 2\xi^4\rho - 2\xi^4\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) \end{aligned}$$

ρ : Quantum fluid density
$\mathbf{v}$: Quantum fluid velocity

Quantized vortex line ($\rho = 0$)

$$\begin{aligned} \omega &= \nabla\times\mathbf{v} = \mathbf{0} \quad (\text{for } \rho \neq 0) \\ \int_C d\mathbf{l}\cdot\mathbf{v} &= (2\pi n)2\xi^2 \quad (n = 0, \pm1, \pm2\cdots) \end{aligned}$$



3 Numerical simulation

3.1 Dissipation and Forcing

GP equation (in wave vector space)

$$i\frac{\partial}{\partial t}\psi_{\mathbf{k}} = \xi^2 k^2 \psi_{\mathbf{k}} - \psi_{\mathbf{k}} + \int d\mathbf{p} d\mathbf{q} d\mathbf{r} \delta(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r}) \psi^*_{\mathbf{p}} \psi_{\mathbf{q}} \psi_{\mathbf{r}}$$
$$- i\nu k^2 \psi_{\mathbf{k}} + i\alpha_{\mathbf{k}} \psi_{\mathbf{k}}$$

- Dissipation
  - The dissipation term acts mainly in the high wavenumber range ($k \sim> 1/\xi$).
- Forcing (Pumping of condensates)

$$\alpha_{\mathbf{k}} = \begin{cases} \alpha & (k < k_f) \\ 0 & (k \geq k_f) \end{cases}$$

  - α is determined at every time step so as to keep $\bar{\rho}$ almost constant.

3.2 Simulation conditions

- $(2\pi)^3$ box with periodic boundary conditions.
- An alias-free spectral method with a Fast Fourier Transform.
- A 4th order Runge-Kutta method for time marching.
- Resolution $k_{\max}\xi = 3$.
- $\nu = \xi^2$.

N	$k_{\max}$	ξ	$\nu(\times 10^{-3})$	k_f	Δt	$\bar{\rho}$
128	60	0.05	2.5	2.5	0.01	0.998
256	120	0.025	0.625	2.5	0.01	0.999
512	241	0.0125	0.15625	2.5	0.01	0.998

3.3 Energy

Energy density per unit volume

$$E = E^{\mathrm{kin}} + E^{\mathrm{int}}$$

$$E^{\mathrm{kin}} = \frac{1}{V}\int d\mathbf{x}\xi^2|\nabla\psi|^2 = \int d\mathbf{k}\xi^2 k^2|\psi_{\mathbf{k}}|^2 = \int dk E^{\mathrm{kin}}(k)$$

$$E^{\mathrm{int}} = \frac{1}{2V}\int d\mathbf{x}(\rho')^2 = \frac{1}{2}\int d\mathbf{x}|\rho'_{\mathbf{k}}|^2 = \int dk E^{\mathrm{int}}(k) \qquad (\rho' = \rho - \bar{\rho})$$

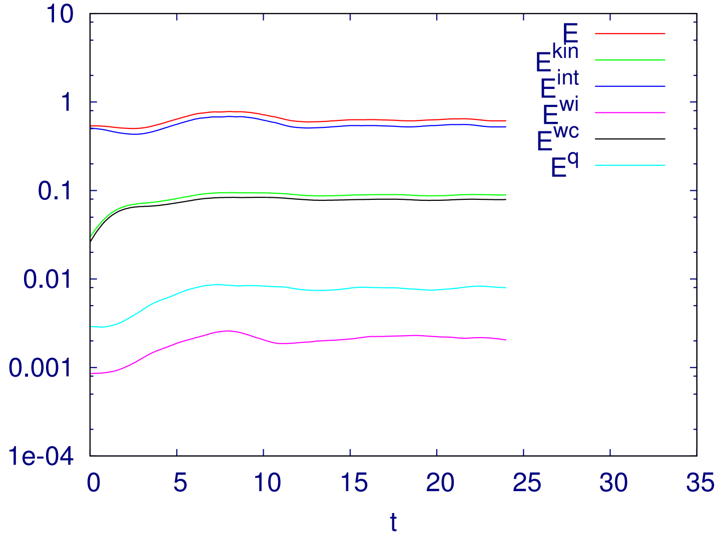
$$E^{\mathrm{kin}} = E^{\mathrm{wi}} + E^{\mathrm{wc}} + E^{\mathrm{q}}$$

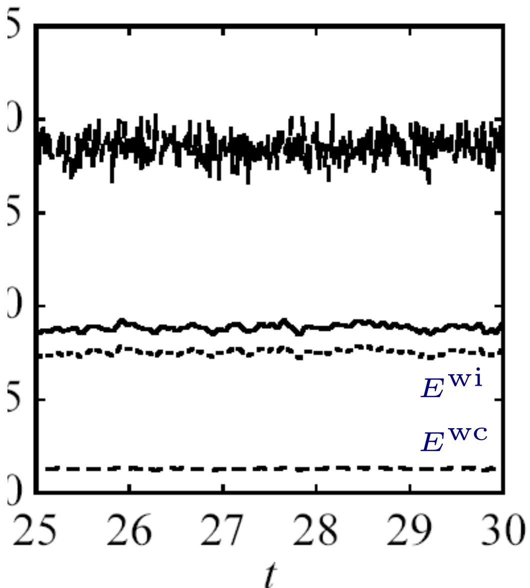
$$E^{\mathrm{wi}} = \frac{1}{2V}\int d\mathbf{x}|\mathbf{w}^{\mathrm{i}}|^2 = \frac{1}{2}\int d\mathbf{k}|\mathbf{w}^{\mathrm{i}}_{\mathbf{k}}|^2 = \int dk E^{\mathrm{wi}}(k) \qquad \left(\mathbf{w} = \frac{1}{\sqrt{2}\xi}\sqrt{\rho}\mathbf{v}\right)$$

$$E^{\mathrm{wc}} = \frac{1}{2V}\int d\mathbf{x}|\mathbf{w}^{\mathrm{c}}|^2 = \frac{1}{2}\int d\mathbf{k}|\mathbf{w}^{\mathrm{c}}_{\mathbf{k}}|^2 = \int dk E^{\mathrm{wc}}(k)$$

$$E^{\mathrm{q}} = \frac{1}{V}\int d\mathbf{x}\xi^2|\nabla\sqrt{\rho}|^2 = \int d\mathbf{k}\xi^2 k^2|(\sqrt{\rho})_{\mathbf{k}}|^2 = \int dk E^{\mathrm{q}}(k)$$

3.4 Energy in the simulation

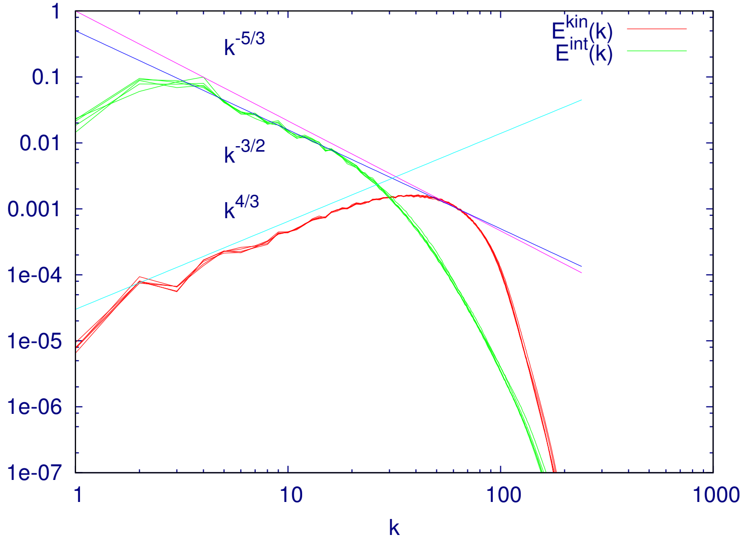


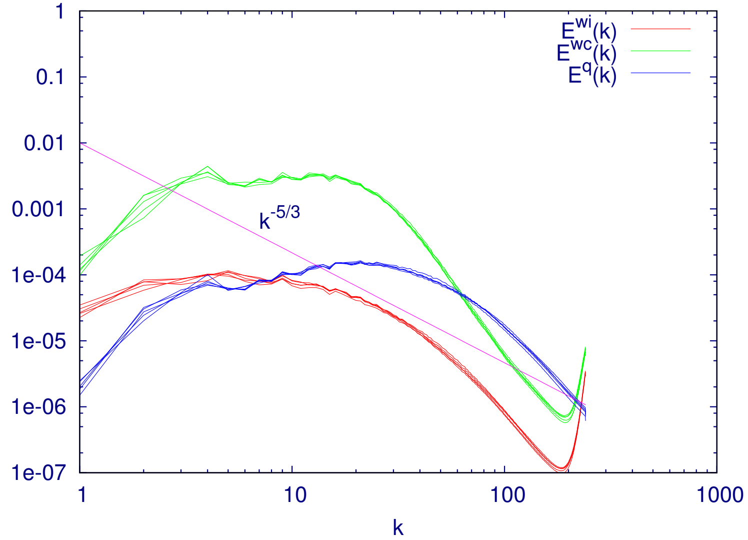


Kobayashi and Tsubota (J. Phys. Soc. Jpn. **57**, 3248(2005))

- $E^{\rm wc} > E^{\rm wi}$. Different from KT.
- Dissipation and forcing are different from those of KT.

3.5 Energy spectrum





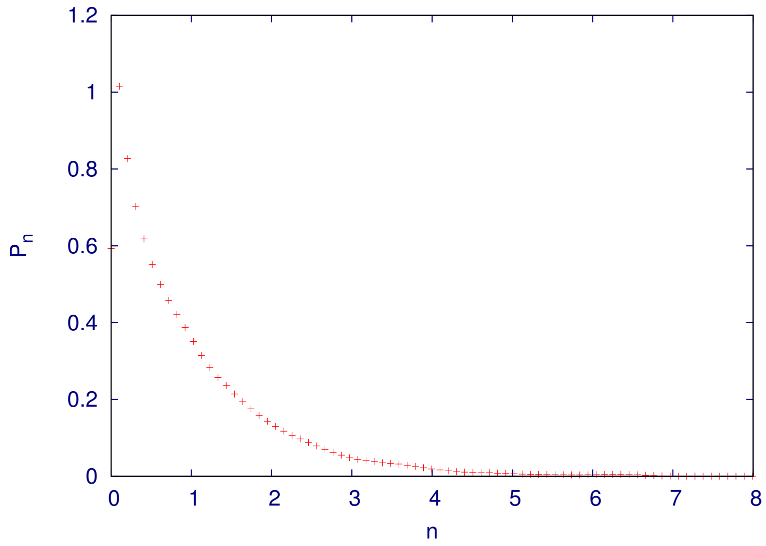
- $E^{\mathrm{int}} \sim k^{-3/2}$.
  - Consistent with the weak turbulence theory.
    (Dyachenko *et. al.* Physica D **57** 96 (1992))
- $E^{\mathrm{kin}} \sim k^{4/3}$.
- $E^{\mathrm{wi}} \sim k^{-5/3}$ is not observed.
  - $E^{\mathrm{wi}} \sim k^{-5/3}$ is observed in KT. Difference in the forcings?

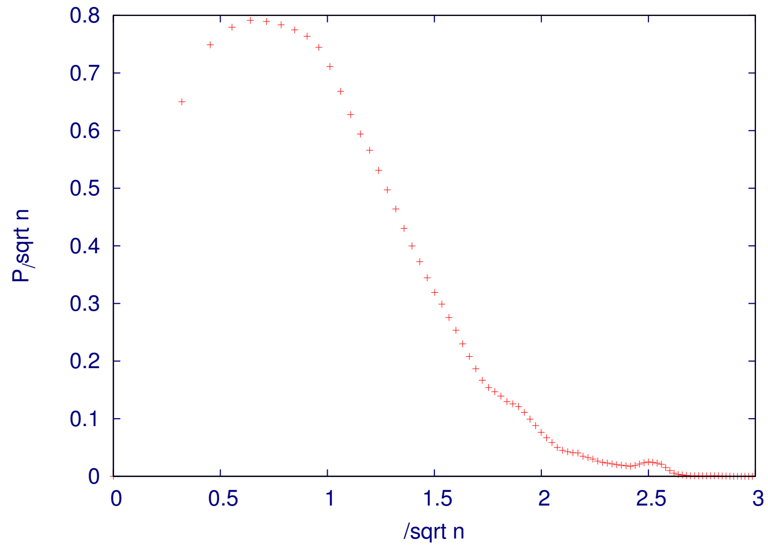
3.6 PDF of the density field

$$\rho(\mathbf{x},t) = |\psi(\mathbf{x},t)|^2, \qquad \sqrt{\rho(\mathbf{x},t)} = |\psi(\mathbf{x},t)|$$

In the weak turbulence theory,

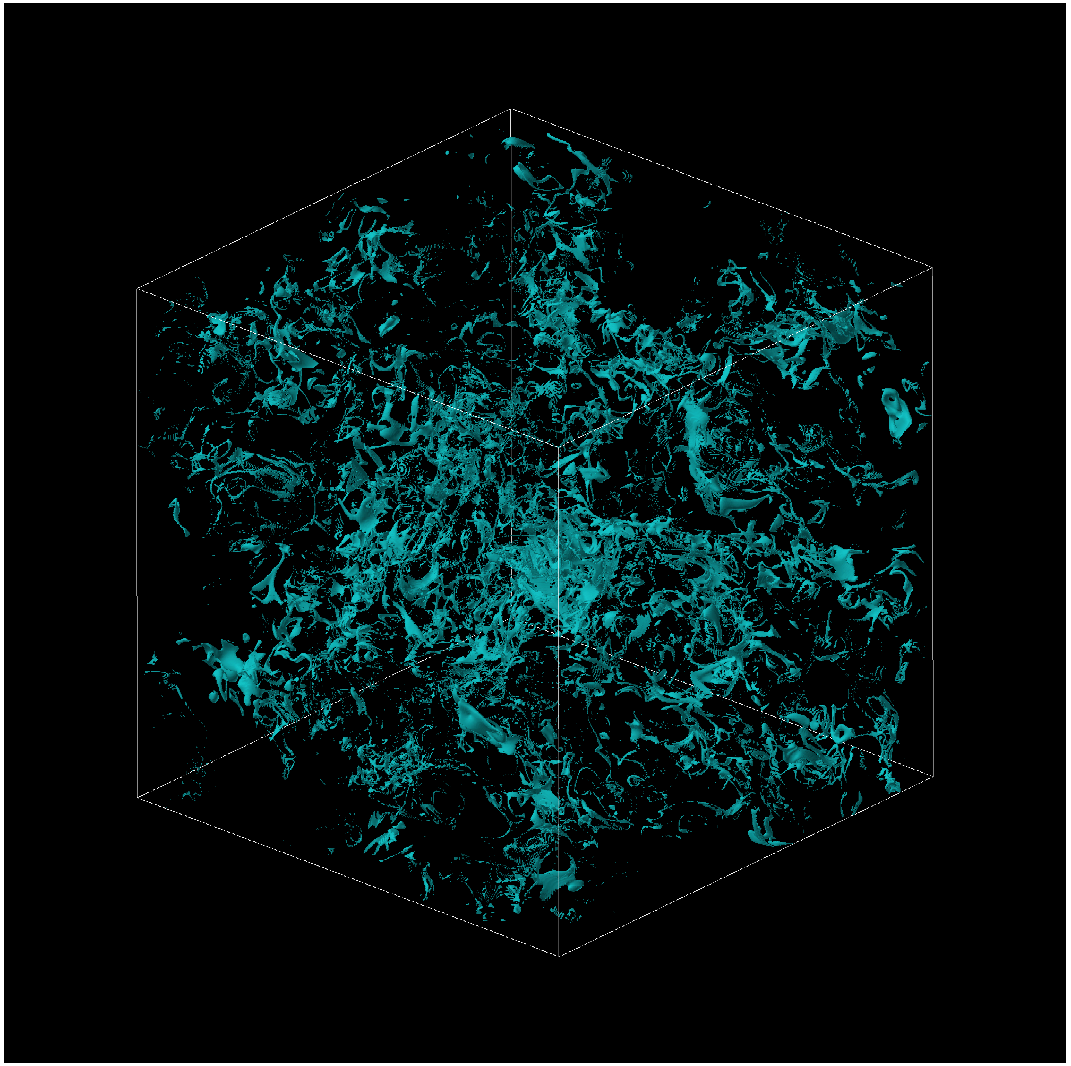
$$\rho(\mathbf{x},t) = \overline{\rho} + \delta\rho(\mathbf{x},t), \qquad |\delta\rho| \ll \overline{\rho}.$$





The turbulence is not weak?

3.7 Low density region



$N = 512$

$\xi = 0.0125$

$\rho < 0.0025$

4 Frequency spectrum

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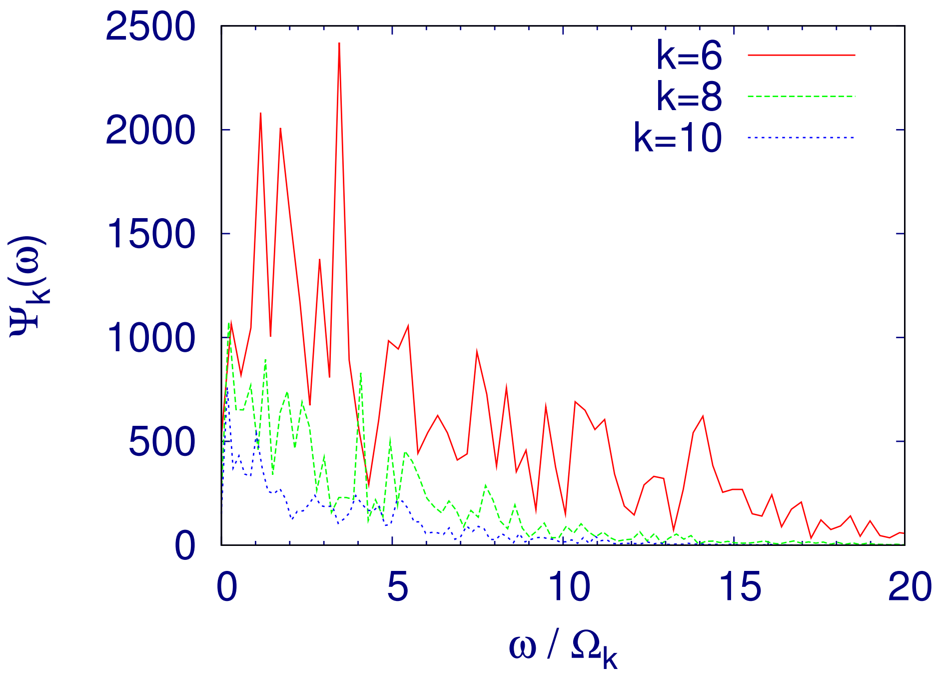
$$\Psi_{\mathbf{k}}(\omega) := \begin{cases} |\psi_{\mathbf{k},\omega}|^2 + |\psi_{\mathbf{k},-\omega}|^2 & (\omega \neq 0) \\ |\psi_{\mathbf{k},\omega}|^2 & (\omega = 0) \end{cases}, \qquad \psi_{\mathbf{k},\omega} := \frac{1}{2\pi} \int_{t_0}^{t_0+T} dt\, \psi_{\mathbf{k}}(t) e^{-i\omega(t-t_0)}.$$

In the weak turbulence theory, it is assumed that

$$\Psi_{\mathbf{k}}(\omega) \sim \delta(\omega - \Omega_k), \qquad \Omega_k := \xi k \sqrt{2 + \xi^2 k^2}.$$



The assumption is not satisfied, *i.e.*, the turbulence is not weak.

5 Summary

Numerical simulations of Gross-Pitaevskii equation with forcing and dissipation are performed up to 512^3 grid points.

- $E^{\rm int}(k) \sim k^{-3/2}$.
  - The scaling coincides with that in the weak turbulence theory. However, it is found that the turbulence is not weak, *i.e.*, $|\delta\rho| \sim \overline{\rho}$ and $\Psi_{\mathbf{k}}(\omega) \neq \delta(\omega - \Omega_k)$.
  - A possible scenario for the explanation of the scaling is to introduce the time scale of decorrelation $\tau(k) \sim \Omega_k^{-1}$. Closure analysis (DIA, LRA)?
- $E^{\rm wi}(k) \sim k^{-5/3}$ is not so clearly observed.
  - The present result is different from that in Kobayashi and Tsubota (2005). Presumably, the forcing in the present simulation injects little to $E^{\rm wi}$ of the system.