§1 Introduction

Kraichnan, J. Fluid Mech. 5, 497-543 (1959) (DIA)
Kraichnan, Phys. Fluids 8, 575-598 (1965) (ALHDIA)
Kraichnan and Herring, J. Fluid Mech 88 355-367 (1978) (SBALHDIA)
Kaneda, J. Fluid Mech 107 131-145 (1981) (LRA).

Leslie, 'Developments in the theory of turbulence'
(Clarendon Press, Oxford, 1973)
Kaneda Fluid Dynamics Research 39 526-551 (2007) (Review)
後藤俊幸「乱流理論の基礎」(朝倉書店, 1992)

Incompressible Navier-Stokes equations

$$\frac{\partial u^\alpha}{\partial t} + u^\beta \frac{\partial}{\partial x^\beta} u^\alpha = -\frac{\partial p}{\partial x^\alpha} + \nu \frac{\partial^2 u^\alpha}{\partial x^\beta \partial x^\beta} + f^\alpha \tag{1}$$

$$\frac{\partial u^\alpha}{\partial x^\alpha} = 0 \tag{2}$$

$u^\alpha(\underset{\sim}{x}, t)$: velocity field
$p(\underset{\sim}{x}, t)$: pressure field

ν : kinematic viscosity
f^α : external force field

Fourier Transform

$$\begin{cases} u^\alpha(x) = \int_{\underset{\sim}{k}} u^\alpha_{\underset{\sim}{k}} e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} & (3) \\ u^\alpha_{\underset{\sim}{k}} = \frac{1}{(2\pi)^d}\int_{\underset{\sim}{x}} u^\alpha(\underset{\sim}{x})\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}} & (4) \end{cases}$$

Navier-Stokes equations in Fourier Space

$$\begin{cases} \frac{\partial}{\partial t} u^\alpha_{\underset{\sim}{k}} = \int_{\underset{\sim}{p},\underset{\sim}{q}} \delta(\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q})\, M^{\alpha\beta\gamma}_{\underset{\sim}{k}}\, u^\beta_{\underset{\sim}{p}}\, u^\gamma_{\underset{\sim}{q}} \\ \qquad\qquad - \nu k^2 u^\alpha_{\underset{\sim}{k}} + D^{\alpha\gamma}_{\underset{\sim}{k}} f^\gamma_{\underset{\sim}{k}} & (5) \\ \underset{\sim}{k}^\alpha u^\alpha_{\underset{\sim}{k}} = 0 & (6) \end{cases}$$

$$M^{\alpha\beta\gamma}_{\underset{\sim}{k}} := -\frac{i}{2}\left(k^\beta D^{\alpha\gamma}_{\underset{\sim}{k}} + k^\gamma D^{\alpha\beta}_{\underset{\sim}{k}}\right) \qquad (7)$$

$$D^{\alpha\beta}_{\underset{\sim}{k}} := \delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2} \qquad (8)$$

reality condition

$$u^\alpha_{-\underset{\sim}{k}} = u^{\alpha\,*}_{\underset{\sim}{k}} \qquad (9)$$

$$\frac{\partial}{\partial t}\frac{1}{2}|u^{\alpha}_{\mathbf{k}}|^2 = \frac{1}{2}\int_{\mathbf{p},\mathbf{q}} \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\left(\tilde{H}^{\alpha\alpha}_{\mathbf{k}\mathbf{p}\mathbf{q}} + \overset{\text{c.c.}}{\tilde{H}^{\alpha\alpha}_{-\mathbf{k}-\mathbf{p}-\mathbf{q}}}\right)$$

$$-2\nu k^2 |u^{\alpha}_{\mathbf{k}}|^2 + D^{\alpha\gamma}_{\mathbf{k}} f^{\gamma}_{\mathbf{k}} u^{\alpha}_{-\mathbf{k}} + D^{\alpha\gamma}_{\mathbf{k}} u^{\alpha}_{\mathbf{k}} f^{\gamma}_{\mathbf{k}} \qquad \text{—(10)}$$

$$\tilde{H}^{\alpha\beta}_{\mathbf{k}\mathbf{p}\mathbf{q}} := M^{\alpha\gamma\zeta}_{\mathbf{k}}\, u^{\gamma}_{\mathbf{p}}\, u^{\zeta}_{\mathbf{q}}\, u^{\beta}_{-\mathbf{k}} \qquad \text{— (11)}$$

$$-\frac{1}{2}\left(\tilde{H}^{\alpha\alpha}_{\mathbf{k}\mathbf{p}\mathbf{q}} + \tilde{H}^{\alpha\alpha}_{-\mathbf{k}-\mathbf{p}-\mathbf{q}}\right) \qquad \text{—(12)}$$

energy transfer rate $\mathbf{k} \to \mathbf{p},\mathbf{q}$

$$\int_{\mathbf{k}\,|\mathbf{k}|>K} (10)$$

$$\frac{\partial}{\partial t}\frac{1}{2}\int_{\mathbf{k},\,|\mathbf{k}|>K} |u^{\alpha}_{\mathbf{k}}|^2 = \tilde{\Pi}(K) - \tilde{\epsilon}(K) + \tilde{F}(K) \qquad \text{—(13)}$$

- energy flux into $|\mathbf{k}| > K$

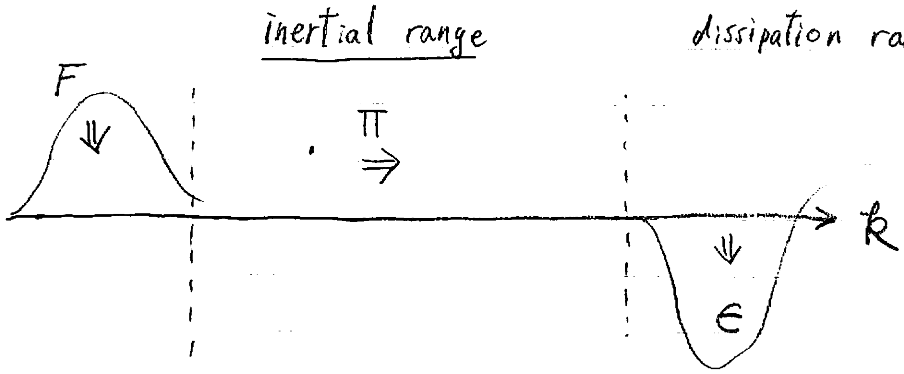
$$\tilde{\Pi}(K) := \frac{1}{2}\int_{\mathbf{k},\,|\mathbf{k}|>K}\int_{\mathbf{p},\mathbf{q}} \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\left(\tilde{H}^{\alpha\alpha}_{\mathbf{k}\mathbf{p}\mathbf{q}} + \tilde{H}^{\alpha\alpha}_{-\mathbf{k}-\mathbf{p}-\mathbf{q}}\right) \qquad \text{—(14)}$$

- energy dissipation rate in $|\mathbf{k}| > K$

$$\tilde{\epsilon}(K) := \nu\int_{\mathbf{k},\,|\mathbf{k}|>K} k^2 |u^{\alpha}_{\mathbf{k}}|^2 \qquad \text{— (15)}$$

- energy injection rate in $|\mathbf{k}| > K$

$$\tilde{F}(K) := \frac{1}{2}\int_{\mathbf{k},\,|\mathbf{k}|>K}\left(D^{\alpha\gamma}_{\mathbf{k}} f^{\gamma}_{\mathbf{k}} u^{\alpha}_{-\mathbf{k}} + D^{\beta\gamma}_{\mathbf{k}} u^{\alpha}_{\mathbf{k}} f^{\gamma}_{-\mathbf{k}}\right) \qquad \text{—(16)}$$



When K is in the inertial range.

$$\tilde{\epsilon}(K) \approx \tilde{\epsilon}(0) =: \tilde{\epsilon} \qquad \text{—(17)}$$

$$F(K) \approx 0 \qquad \text{—(18)}$$

(13), (17), (18)

$$\frac{\partial}{\partial t}\frac{1}{2}\int_{\underset{\sim}{k},\, |\underset{\sim}{k}|>K} |u^{\alpha}_{\underset{\sim}{k}}|^2 \approx \tilde{\Pi}(K) - \tilde{\epsilon} \qquad \text{—(19)}$$

if

$$\frac{\partial}{\partial t}\frac{1}{2}\int_{\underset{\sim}{k},\, |\underset{\sim}{k}|>K} |u^{\alpha}_{\underset{\sim}{k}}|^2 \approx 0 \qquad (20)$$

$$\tilde{\Pi}(K) \approx \tilde{\epsilon} \qquad (21)$$

Statistically averaged quantities

$$Q^{\alpha\beta}_{\underset{\sim}{k}}(t) := \frac{1}{\delta(\underset{\sim}{0})}\langle u^{\alpha}_{\underset{\sim}{k}}(t)\, u^{\beta}_{-\underset{\sim}{k}}(t)\rangle \quad \text{— (22)}$$

$\frac{1}{\delta(0)} \leftarrow$ Homogeneity in space.

$$H^{\alpha\beta}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t) := \frac{1}{\delta(0)}\langle \widetilde{H}^{\alpha\beta}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t)\rangle$$

$$= \frac{1}{\delta(0)} M^{\alpha\gamma\zeta}_{\underset{\sim}{k}} \langle u^{\gamma}_{\underset{\sim}{p}}(t)\, u^{\zeta}_{\underset{\sim}{q}}(t)\, u^{\beta}_{-\underset{\sim}{k}}(t)\rangle \quad \text{— (23)}$$

$\frac{1}{\delta(0)}\langle (13)\rangle$

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\underset{\sim}{k},\, |\underset{\sim}{k}|>K} Q^{\alpha\alpha}_{\underset{\sim}{k}}(t) = \Pi(K,t) - \epsilon(K,t) + F(K,t) \quad \text{— (24)}$$

$$\Pi(K,t) = \frac{1}{\delta(\underset{\sim}{0})}\langle \widetilde{\Pi}(K,t)\rangle$$

$$= \frac{1}{2}\int_{\underset{\sim}{k},\, |\underset{\sim}{k}|\geq K}\int_{\underset{\sim}{p},\underset{\sim}{q}} \delta(\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q})\left(H^{\alpha\alpha}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}} + H^{\alpha\alpha}_{-\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q}}\right) \quad \text{— (25)}$$

$$\epsilon(K,t) = \frac{1}{\delta(0)}\langle \widetilde{\epsilon}(K,t)\rangle$$

$$= \nu\int_{\underset{\sim}{k},\, |\underset{\sim}{k}|>K} k^2\, Q^{\alpha\alpha}_{\underset{\sim}{k}}(t) \quad \text{— (26)}$$

$$F(K,t) = \frac{1}{\delta(0)}\langle F(K,t)\rangle \quad \text{— (27)}$$

Need a model for $\Pi(K,t)$ or $H^{\alpha\beta}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t)$.

Express $H^{\alpha\beta}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t)$ in terms of $Q^{\alpha\beta}_{\underset{\sim}{k}}(t)$

(and other quantities.)

§2 Derivation of Direct Interaction Approximation

Two-time correlation function

↙ statistical homegeneity in space

$$\langle u_{k}^{\alpha}(t)\, u_{-k'}^{\beta}(s)\rangle = Q_{k}^{\alpha\beta}(t,s)\,\delta(k-k') \tag{1}$$

Two-time response function

$$\langle \tilde{G}_{kk'}^{\alpha\beta}(t,s)\rangle = G_{k}^{\alpha\beta}(t,s)\,\delta(k-k') \tag{2}$$

$$G_{kk'}^{\alpha\beta} := \frac{\delta u_{k}^{\alpha}(t)}{\delta f_{k'}^{\beta}(t)} \tag{3}$$

(1.5) $M \to \lambda M \quad (\lambda = 1)$

$$\frac{\partial}{\partial t} u_{k}^{\alpha} = \lambda \int_{p,q} \delta(k-p-q)\, M_{k}^{\alpha\beta\gamma} u_{p}^{\beta} u_{q}^{\gamma} - \nu k^2 u_{k}^{\alpha} + D_{k}^{\alpha\gamma} f_{k}^{\gamma} \tag{4}$$

$f = 0$

$$\frac{\partial}{\partial t} Q_{k}^{\alpha\beta}(t,t) = \int_{p,q} \delta(k-p-q)\left[H_{kpq}^{\alpha\beta}(t) + H_{-k-p-q}^{\beta\alpha}(t)\right] - 2\nu k^2 Q_{k}^{\alpha\beta}(t,t) \tag{5}$$

$$\frac{\partial}{\partial t} Q_{k}^{\alpha\beta}(t,s) = \int_{p,q} \delta(k-p-q)\, I_{kpq}^{\alpha\beta}(t,s) - \nu k^2 Q_{k}^{\alpha\beta}(t,s) \tag{6}$$

$$\frac{\partial}{\partial t} G_{k}^{\alpha\beta}(t,s) = \int_{p,q} \delta(k-p-q)\, J_{kpq}^{\alpha\beta}(t,s) - \nu k^2 G_{k}^{\alpha\beta}(t,s) \tag{7}$$

$$G_{k}^{\alpha\beta}(t,t) = D_{k}^{\alpha\beta} \tag{8}$$

$$H^{\alpha\beta}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t) := \lambda \frac{1}{\delta(0)} M^{\alpha\gamma\zeta}_{\underset{\sim}{k}} \langle u^{\gamma}_{\underset{\sim}{p}}(t)\, u^{\zeta}_{\underset{\sim}{q}}(t)\, u^{\beta}_{-\underset{\sim}{k}}(t) \rangle \tag{9}$$

$$I^{\alpha\beta}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t,s) := \lambda \frac{1}{\delta(0)} M^{\alpha\gamma\zeta}_{\underset{\sim}{k}} \langle u^{\gamma}_{\underset{\sim}{p}}(t)\, u^{\zeta}_{\underset{\sim}{q}}(t)\, u^{\beta}_{-\underset{\sim}{k}}(t) \rangle \tag{10}$$

$$J^{\alpha\beta}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t,s) := \lambda \frac{2}{\delta(0)} M^{\alpha\gamma\zeta}_{\underset{\sim}{k}} \langle \tilde{G}^{\gamma\beta}_{\underset{\sim}{p}\underset{\sim}{k}}(t,s)\, u^{\zeta}_{\underset{\sim}{q}}(t) \rangle \tag{11}$$

$\boxed{\text{Purpose} \quad \text{approximate } H, I, J \text{ in terms of } Q, G.}$

Primitive expansion in λ

$$u^{\alpha}_{\underset{\sim}{k}} = u^{\alpha(0)}_{\underset{\sim}{k}} + \lambda u^{\alpha(1)}_{\underset{\sim}{k}} + \lambda^2 u^{\alpha(2)}_{\underset{\sim}{k}} + \cdots \tag{12}$$

$$\tilde{G}^{\alpha\beta}_{\underset{\sim}{k}\underset{\sim}{k}'} = \tilde{G}^{\alpha\beta(0)}_{\underset{\sim}{k}\underset{\sim}{k}'} + \lambda \tilde{G}^{\alpha\beta(1)}_{\underset{\sim}{k}\underset{\sim}{k}'} + \lambda^2 \tilde{G}^{\alpha\beta(2)}_{\underset{\sim}{k}\underset{\sim}{k}'} + \cdots \tag{13}$$

From (7), (8), (13)

$$G^{\alpha\beta(0)}_{\underset{\sim}{k}\underset{\sim}{k}'}(t,s) = \begin{cases} e^{-\nu k^2 (t-s)} D^{\alpha\beta}_{\underset{\sim}{k}}\, \delta(\underset{\sim}{k}-\underset{\sim}{k}') & (t \geq s) \\ 0 & (t < s) \end{cases} \tag{14}$$

From (4), (14)

$$u^{\alpha}_{\underset{\sim}{k}}(t) = u^{\alpha(0)}_{\underset{\sim}{k}}(t) + \lambda \int_{t_0}^{t} dt' \int_{\underset{\sim}{k}'} \tilde{G}^{\alpha\beta(0)}_{\underset{\sim}{k}\underset{\sim}{k}'}(t,t') \int_{\underset{\sim}{p}',\underset{\sim}{q}'} \delta(\underset{\sim}{k}'-\underset{\sim}{p}'-\underset{\sim}{q}') M^{\beta\gamma\zeta}_{\underset{\sim}{k}'} u^{\gamma}_{\underset{\sim}{p}'}(t')\, u^{\zeta}_{\underset{\sim}{q}'}(t') \tag{15}$$

$$u^{\alpha(0)}_{\underset{\sim}{k}}(t) = \int_{\underset{\sim}{k}'} \tilde{G}^{\alpha\beta(0)}_{\underset{\sim}{k}\underset{\sim}{k}'}(t,t_0)\, u^{\beta}_{\underset{\sim}{k}'}(t_0) \tag{16}$$

$$\frac{\delta}{\delta f^{\beta}_{\underset{\sim}{k}'}(s)} \; (15) \quad (t_0 \to s)$$

$$\begin{aligned}\tilde{G}^{\alpha\beta}_{\underset{\sim}{k}\underset{\sim}{k}'}(t,s) = {} & \tilde{G}^{\alpha\beta}_{\underset{\sim}{k}\underset{\sim}{k}'}(t,s) \\ & + 2\lambda \int_s^t dt' \int_{\underset{\sim}{k}''} G^{\alpha\eta\,(0)}_{\underset{\sim}{k}\underset{\sim}{k}''}(t,t') \int_{\underset{\sim}{p}'',\underset{\sim}{q}''} \delta(\underset{\sim}{k}''-\underset{\sim}{p}''-\underset{\sim}{q}'') \\ & \times M^{\eta\gamma\zeta}_{\underset{\sim}{k}''} \tilde{G}^{\gamma\beta}_{\underset{\sim}{p}''\underset{\sim}{k}'}(t',s)\, u^{\zeta}_{\underset{\sim}{q}''}(t')\end{aligned} \tag{17}$$

Assumption

Odd moments

$$\langle u^{\alpha(0)}_{\underset{\sim}{k}} \rangle = 0 \quad , \quad \langle u^{\alpha(0)}_{\underset{\sim}{k}} u^{\beta(0)}_{\underset{\sim}{p}} u^{\gamma}_{\underset{\sim}{q}} \rangle = 0 \quad , \quad \cdots \tag{18}$$

$$\begin{aligned}\langle u^{\alpha(0)}_{\underset{\sim}{k}} u^{\beta(0)}_{\underset{\sim}{p}} u^{\gamma(0)}_{\underset{\sim}{q}} u^{\zeta}_{\underset{\sim}{r}} \rangle = {} & \langle u^{\alpha(0)}_{\underset{\sim}{k}} u^{\beta(0)}_{\underset{\sim}{p}} \rangle \langle u^{\gamma(0)}_{\underset{\sim}{q}} u^{\zeta(0)}_{\underset{\sim}{r}} \rangle \\ & + \langle u^{\alpha(0)}_{\underset{\sim}{k}} u^{\gamma(0)}_{\underset{\sim}{q}} \rangle \langle u^{\beta(0)}_{\underset{\sim}{p}} u^{\zeta(0)}_{\underset{\sim}{r}} \rangle \\ & + \langle u^{\alpha(0)}_{\underset{\sim}{k}} u^{\zeta(0)}_{\underset{\sim}{r}} \rangle \langle u^{\beta(0)}_{\underset{\sim}{p}} u^{\gamma(0)}_{\underset{\sim}{q}} \rangle\end{aligned} \tag{19}$$

Primitive expansion

$$Q_{\underset{\sim}{k}}^{\alpha\beta}(t,s) = Q_{\underset{\sim}{k}}^{\alpha\beta\,(0)}(t,s) + \lambda\, Q_{\underset{\sim}{k}}^{\alpha\beta\,(1)}(t,s)\left[Q^{(0)}, G^{(0)}\right] + O(\lambda^2) \tag{20}$$

$$G_{\underset{\sim}{k}}^{\alpha\beta}(t,s) = G_{\underset{\sim}{k}}^{\alpha\beta\,(0)}(t,s) + \lambda\, G_{\underset{\sim}{k}}^{\alpha\beta\,(1)}(t,s)\left[Q^{(0)}, G^{(0)}\right] + O(\lambda^2) \tag{21}$$

$$H_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}^{\alpha\beta}(t) = \lambda^2 H_{\underset{\sim}{k}\underset{\sim}{l}\underset{\sim}{q}}^{\alpha\beta\,(2)}(t)\left[Q^{(0)}, G^{(0)}\right] + O(\lambda^3) \tag{22}$$

$$I_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}^{\alpha\beta}(t,s) = \lambda^2 I_{\underset{\sim}{k}\underset{\sim}{l}\underset{\sim}{q}}^{\alpha\beta\,(2)}(t,s)\left[Q^{(0)}, G^{(0)}\right] + O(\lambda^3) \tag{23}$$

$$J_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}^{\alpha\beta}(t,s) = \lambda^2 J_{\underset{\sim}{k}\underset{\sim}{l}\underset{\sim}{q}}^{\alpha\beta\,(2)}(t,s)\left[Q^{(0)}, G^{(0)}\right] + O(\lambda^3) \tag{24}$$

Inverse expansion

$$Q_{\underset{\sim}{k}}^{\alpha\beta\,(0)}(t,s) = Q_{\underset{\sim}{k}}^{\alpha\beta}(t,s) + \lambda\, Q_{\underset{\sim}{k}}^{\alpha\beta\,[1]}(t,s)\left[Q, G\right] + O(\lambda^2) \tag{25}$$

$$G_{\underset{\sim}{k}}^{\alpha\beta\,(0)}(t,s) = G_{\underset{\sim}{k}}^{\alpha\beta}(t,s) + \lambda\, G_{\underset{\sim}{k}}^{\alpha\beta\,[1]}(t,s)\left[Q, G\right] + O(\lambda^2) \tag{26}$$

Renormalized expansion

(22) ← (25), (26)

$$H^{\alpha\beta}_{\underset{\sim}{k}\underset{\sim}{l}\underset{\sim}{q}}(t) = \lambda^2 H^{\alpha\beta(2)}_{\underset{\sim}{k}\underset{\sim}{l}\underset{\sim}{q}}(t)[Q, G] + O(\lambda^3) \tag{27}$$

(23) ← (25), (26)

$$I^{\alpha\beta}_{\underset{\sim}{k}\underset{\sim}{l}\underset{\sim}{q}}(t, s) = \lambda^2 I^{\alpha\beta(2)}_{\underset{\sim}{k}\underset{\sim}{l}\underset{\sim}{q}}(t, s)[Q, G] + O(\lambda^3) \tag{28}$$

(24) ← (25), (26)

$$J^{\alpha\beta}_{\underset{\sim}{k}\underset{\sim}{l}\underset{\sim}{q}}(t, s) = \lambda^2 J^{\alpha\beta(2)}_{\underset{\sim}{k}\underset{\sim}{l}\underset{\sim}{q}}(t, s)[Q, G] + O(\lambda^3) \tag{29}$$

Truncation

Retain the non-trivial leading order terms.

i.e. $O(\lambda^2)$

$\Downarrow$

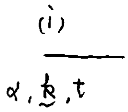
Direct Interaction Approximation (DIA).

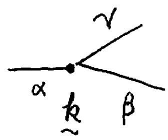
H, I, J are expressed in terms of Q, G.

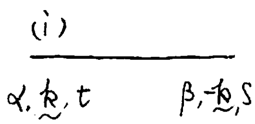
Thus Eqs. (5) – (8) are closed.

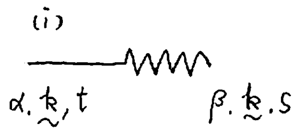
Diagram expression

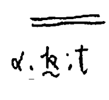
(0) will be omitted.

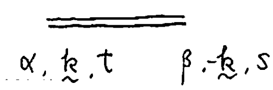
$\mathcal{U}^{\alpha\,(i)}_{\underset{\sim}{k}}(t)$ $\longleftrightarrow$ 

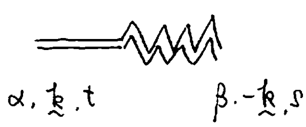
$M^{\alpha\beta\gamma}_{\underset{\sim}{k}}$ $\longleftrightarrow$ 

$Q^{\alpha\beta\,(i)}_{\underset{\sim}{k}}(t,s)$ $\longleftrightarrow$ 

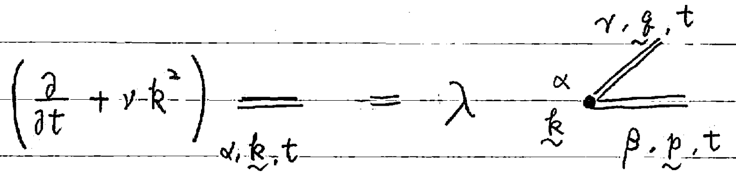
$G^{\alpha\beta\,(i)}_{\underset{\sim}{k}}(t,s)$ $\longleftrightarrow$ 

$\mathcal{U}^{\alpha}_{\underset{\sim}{k}}(t)$ $\longleftrightarrow$ 

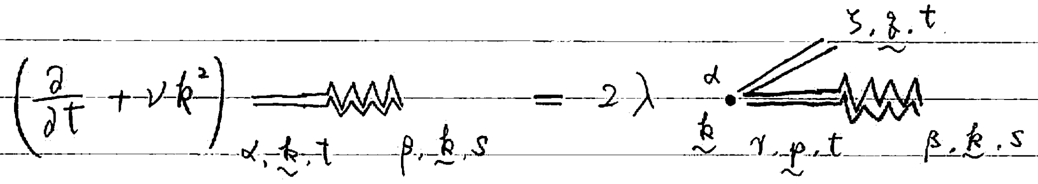
$Q^{\alpha\beta}_{\underset{\sim}{k}}(t,s)$ $\longleftrightarrow$ 

$G^{\alpha\beta}_{\underset{\sim}{k}}(t,s)$ $\longleftrightarrow$ 

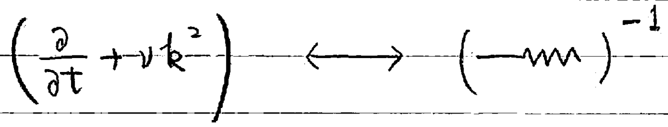
(30)



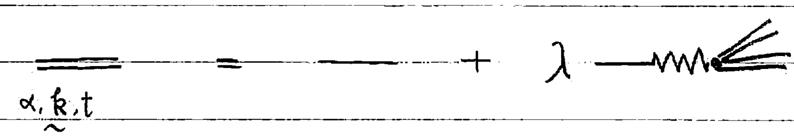
(31) ⟷ (4)



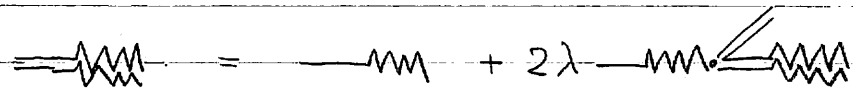
(32) ⟷ (7)
before taking ⟨ ⟩



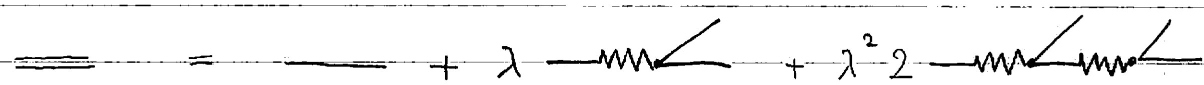
(33)



(34) ⟷ (15)

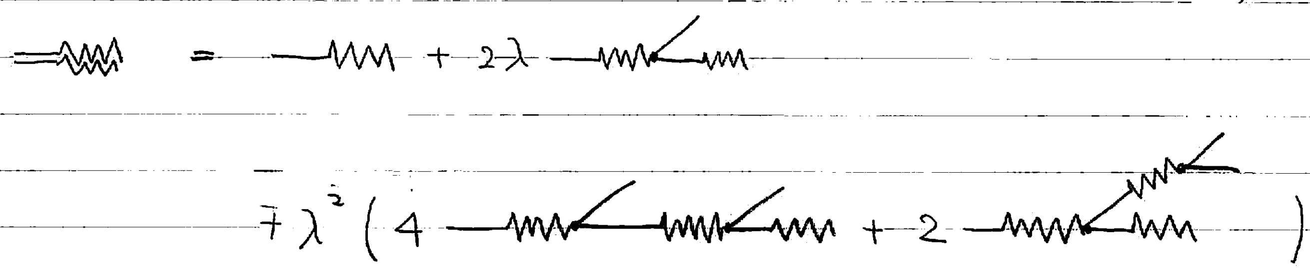


(35) ⟷ (17)



(36)

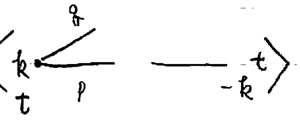
$+ O(\lambda^3)$



$+ O(\lambda^3)$ (37)

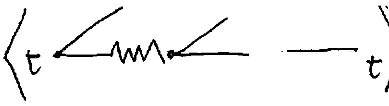
$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) Q^{\alpha\beta}_{\underset{\sim}{k}}(t) \qquad \left(H^{\alpha\beta}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t)\right)$$

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) \overline{\overline{}}_{\alpha,\underset{\sim}{k},t \quad \beta,-\underset{\sim}{k},t} = \lambda \left(\left\langle \; \right. \right.$$



$$\left. \left. \; \right\rangle + c.c \right)$$

$$+ \lambda^2 \left(2 \left\langle \; \right. \right.$$



$$\left. \; \right\rangle$$

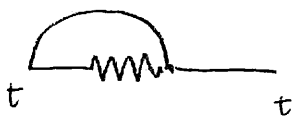
$$+ \left\langle \; \right.$$

$$\left. \; \right\rangle$$

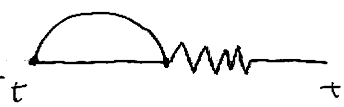
$$\left. + c.c \right)$$

$$+ O(\lambda^3)$$

$$= \lambda^2 \left(4 \right.$$



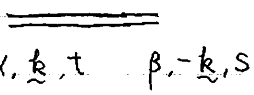
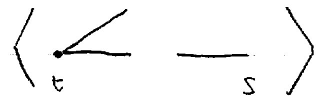
$$+ 2$$

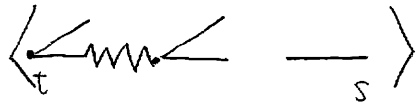


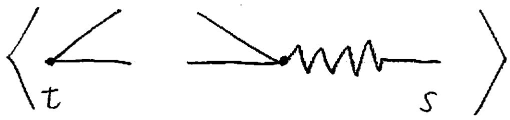
$$\left. + c.c. \right)$$

$$+ O(\lambda^3) \qquad (38)$$

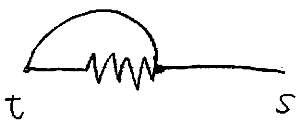
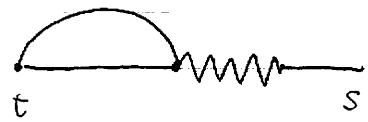
$$\left(\frac{\partial}{\partial t} + \nu k^2\right) Q_{\underset{\sim}{k}}^{\alpha\beta}(t,s) \qquad \left(I_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}^{\alpha\beta}(t,s)\right)$$

$\left(\frac{\partial}{\partial t} + \nu k^2\right)$  $= \lambda$ 

$+ \lambda^2 \Big(2$ 

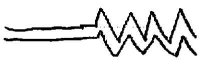
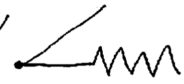
$+$  $\Big)$

$+ O(\lambda^3)$

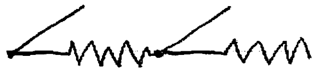
$= \lambda^2 \Big(4$  $+ 2$  $\Big)$

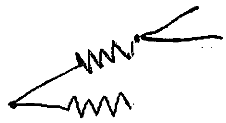
$+ O(\lambda^3)$ (39)

$\left(\frac{\partial}{\partial t}+\nu k^2\right) G^{\alpha\beta}_{k}(t,s) \qquad \left(J^{\alpha\beta}_{klq}(t,s) \right)$

$\left(\frac{\partial}{\partial t}+\nu k^2\right)$  $= \lambda \left(2 \left\langle \right.\right.$  $\left.\left.\right\rangle \right)$

$\alpha, k, t \qquad \beta, k, s$

$+ \lambda^2 \Big(4$ 

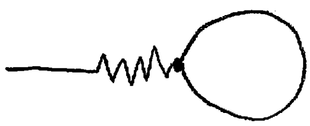
$+ 2$  $\Big)$

$+ O(\lambda^3)$

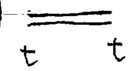
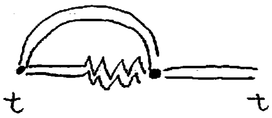
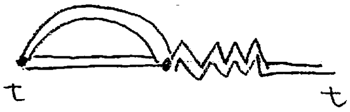
$= \lambda^2 \, 4$ 

$+ O(\lambda^3) \qquad (40)$

Note

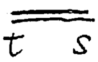
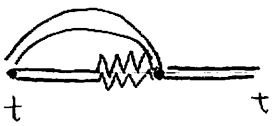
 $= 0 \qquad (41)$

Renormalized expansion

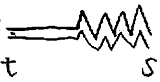
$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)$$  $$= \lambda^2 \Big(4$$  $$+\ 2$$ 

$$+\ \mathrm{c.c.} \Big)$$

$$+\ O(\lambda^3) \qquad (42)$$

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)$$  $$= \lambda^2 \Big(4$$  $$+\ 2$$  $$\Big)$$

$$+\ O(\lambda^3) \qquad (43)$$

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)$$  $$= \lambda^2\ 4$$ 

$$+\ O(\lambda^3) \qquad (44).$$

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right) Q_{\underset{\sim}{k}}^{\alpha\beta}(t) = \lambda^2\int_{\underset{\sim}{p},\underset{\sim}{q}} \delta(\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q}) \int_{t_0}^{t} dt'$$

$$\times\Big[\, 4 M_{\underset{\sim}{k}}^{\alpha\gamma\zeta} M_{\underset{\sim}{p}}^{\eta\theta\kappa} G_{\underset{\sim}{p}}^{\gamma\eta}(t,t')\, Q_{\underset{\sim}{q}}^{\zeta\theta}(t,t')\, Q_{-\underset{\sim}{k}}^{\beta\kappa}(t,t')$$

$$-2 M_{\underset{\sim}{k}}^{\alpha\gamma\zeta} M_{\underset{\sim}{k}}^{\kappa\eta\theta} Q_{\underset{\sim}{p}}^{\gamma\eta}(t,t')\, Q_{\underset{\sim}{q}}^{\zeta\theta}(t,t')\, G_{-\underset{\sim}{k}}^{\beta\kappa}(t,t')\Big]$$

$$+\ (\underset{\sim}{k}\leftrightarrow -\underset{\sim}{k},\ \alpha\leftrightarrow\beta)\ +\ O(\lambda^3) \qquad (45)$$

$$\left(\frac{\partial}{\partial t}+\nu k^2\right) Q_{\underset{\sim}{k}}^{\alpha\beta}(t,s) = \lambda^2\int_{\underset{\sim}{p},\underset{\sim}{q}} \delta(\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q}) \int_{t_0}^{t} dt'$$

$$\times\Big[\, 4 M_{\underset{\sim}{k}}^{\alpha\gamma\zeta} M_{\underset{\sim}{p}}^{\eta\theta\kappa} G_{\underset{\sim}{p}}^{\gamma\eta}(t,t')\, Q_{\underset{\sim}{q}}^{\zeta\theta}(t,t')\, Q_{-\underset{\sim}{k}}^{\beta\kappa}(s,t')$$

$$-2 M_{\underset{\sim}{k}}^{\alpha\gamma\zeta} M_{\underset{\sim}{k}}^{\kappa\eta\theta} Q_{\underset{\sim}{p}}^{\gamma\eta}(t,t')\, Q_{\underset{\sim}{q}}^{\zeta\theta}(t,t')\, G_{-\underset{\sim}{k}}^{\beta\kappa}(s,t')\Big]$$

$$+\ O(\lambda^3) \qquad (46)$$

$$\left(\frac{\partial}{\partial t}+\nu k^2\right) G_{\underset{\sim}{k}}^{\alpha\beta}(t,s) = \lambda^2\int_{\underset{\sim}{p},\underset{\sim}{q}} \delta(\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q}) \int_{s}^{t} dt'$$

$$\times\Big[\, 4 M_{\underset{\sim}{k}}^{\alpha\gamma\zeta} M_{\underset{\sim}{p}}^{\eta\theta\kappa} G_{\underset{\sim}{p}}^{\gamma\eta}(t,t')\, Q_{\underset{\sim}{q}}^{\zeta\theta}(t,t')\, G_{\underset{\sim}{k}}^{\kappa\beta}(t',s)\Big]$$

$$+\ O(\lambda^3) \qquad (47)$$

§3 Properties of DIA

✓ Energy conservation ($\nu = 0$)

✓ When $\nu = 0$, compatible with

Equipartition

$$Q_{\underset{\sim}{k}}^{\alpha\beta}(t,t) = C\, D_{\underset{\sim}{k}}^{\alpha\beta} \qquad \text{—— (1)}$$

and Fluctuation-dissipation Relation

$$Q_{\underset{\sim}{k}}^{\alpha\beta}(t,s) = G_{\underset{\sim}{k}}(t,s)\, Q_{\underset{\sim}{k}}^{\alpha\beta}(s,s) \qquad \text{—— (2)}$$

where

$$G_{\underset{\sim}{k}}^{\alpha\beta}(t,s) = G_{\underset{\sim}{k}}(t,s)\, D_{\underset{\sim}{k}}^{\alpha\beta} \qquad \text{—— (3)}$$

✓ Realizability
i.e. compatible with a Langevin model.

✓ DIA can describe energy cascade

X Energy Spectrum $E(k) \propto \overset{\langle u_i^2 \rangle}{U}\, \epsilon^{1/2} k^{-3/2}$ (4)

Inconsistent with the Kolmogorov spectrum

$$E(k) \propto \epsilon^{2/3} k^{-5/3} \qquad (5)$$

Energy spectrum in inertial range.

Assume Isotropy

$$Q^{\alpha\beta}_{\underset{\sim}{k}}(t,s) = \frac{1}{d-1} Q_k(t,s) D^{\alpha\beta}_{\underset{\sim}{k}} \qquad (6)$$

$$G^{\alpha\beta}_{\underset{\sim}{k}}(t,s) = G_k(t,s) D^{\alpha\beta}_{\underset{\sim}{k}} \qquad (7)$$

It can be shown that $\int_{\underset{\sim}{p},\underset{\sim}{q}}$ integrals in (2.46) and (2.47) diverge as $q \to 0$,

when

$$Q_k(t,t) \propto k^a \qquad (8)$$

with

$$a < -d \quad . \qquad (9)$$

Then, (2.46), (2.47) reduce to

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) G_k(t,s) = - C_d U^2 k^2 \int_s^t dt' \, G_k(t,t') \, G_k(t',s) \qquad (10)$$

$$Q_k(t,s) = G_k(t,s) \, Q_k(s,s) \qquad (11)$$

C_d .. const.

$$U^2 = \frac{1}{d} \int_{\underset{\sim}{q}} Q_q(t,T) \qquad (12)$$

← Large scale quantity
↑
outside inertial range

Similarity solution

$$G_k(t,s) = g(Uk(t-s)) \tag{13}$$

$$\frac{\partial}{\partial\tau} g(\tau) = -C_d \int_0^{\tau} d\tau' \, g(\tau-\tau')\, g(\tau') \tag{14}$$

$$g(0) = 1 \tag{15}$$

$$g(\tau) = \frac{J_1(2\sqrt{C_d}\,\tau)}{\tau} \tag{16}$$

J_1 : Bessel function of the first kind of the order 1.

Energy Flux

(1.25)

$$\Pi(K,t) = \frac{1}{2}\int_{\mathbf{k},\,|\mathbf{k}|>K}\int_{\mathbf{p},\mathbf{q}} \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\left(H^{\alpha\alpha}_{\mathbf{k}\mathbf{p}\mathbf{q}} + H^{\alpha\alpha}_{-\mathbf{k}-\mathbf{p}-\mathbf{q}}\right)$$

$$= \frac{1}{2}\int_{\mathbf{k},\,|\mathbf{k}|>K}\int_{\mathbf{p},\mathbf{q}} \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \int_{t_0}^{t} dt'$$

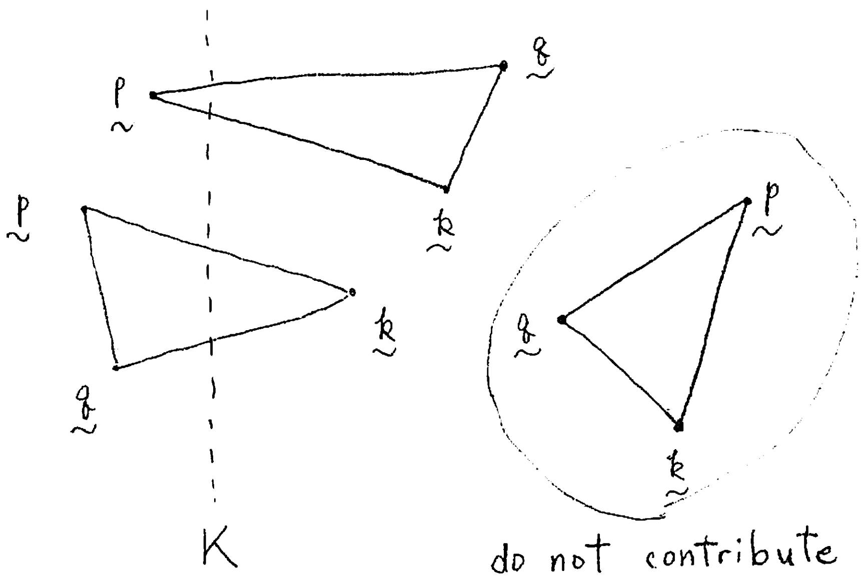
$$\times\Big[4 M^{\alpha\gamma\zeta}_{\mathbf{k}} M^{\eta\theta\kappa}_{\mathbf{p}} G^{\gamma\eta}_{\mathbf{p}}(t,t')\, Q^{\zeta\theta}_{\mathbf{q}}(t,t')\, Q^{\beta\kappa}_{-\mathbf{k}}(t,t')$$

$$-2 M^{\alpha\gamma\zeta}_{\mathbf{k}} M^{\kappa\eta\theta}_{\mathbf{k}} Q^{\gamma\eta}_{\mathbf{p}}(t,t')\, Q^{\zeta\theta}_{\mathbf{q}}(t,t')\, G^{\beta\kappa}_{-\mathbf{k}}(t,t')\Big]$$

$$+ (\mathbf{k}\leftrightarrow -\mathbf{k},\ \alpha\leftrightarrow\beta) \tag{17}$$

Detailed energy balance

$$H^{\alpha\alpha}_{\mathbf{k}\mathbf{p}\mathbf{q}} + H^{\alpha\alpha}_{-\mathbf{p}\,\mathbf{q}-\mathbf{k}} + H^{\alpha\alpha}_{-\mathbf{q}-\mathbf{k}\,\mathbf{p}} = 0 \tag{18}$$



do not contribute to the integral in (17)

$$\Pi(K,t) = - \int_{\underset{\sim}{k},|\underset{\sim}{k}|>K} \int_{\underset{\sim}{p},|\underset{\sim}{p}|>K} \int_{\underset{\sim}{q},|\underset{\sim}{q}|<|\underset{\sim}{p}|} \delta(\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q})$$
$$\times \left(H^{\alpha\alpha}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}} + H^{\alpha\alpha}_{-\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q}} \right) \tag{19}$$

- Assume the similarity form in the inertial range

$$G_k(t,s) = g\left(U k(t-s)\right) \qquad (k_0 \ll k \ll k_1) \tag{20}$$
$$Q_k(t,s) = G_k(t,s)\, Q_k(s,s) \qquad (\quad '' \quad) \tag{21}$$
$$Q_k(t,t) \propto U^b \epsilon^c k^a \qquad (\quad '' \quad) \tag{22}$$

- Assume the $\underset{\sim}{k}, \underset{\sim}{p}, \underset{\sim}{q}$ integral in (19) converge as

$k_0 \to 0$ and $k_1 \to \infty$.

- Assume the constant energy flux

$$\Pi(K) = \epsilon \qquad (k_0 \ll K \ll k_1) \tag{23}$$

Dimensional analysis

$$[\epsilon] = [\Pi(K)] = \left[\int_{k,|k|>K} \int_{p,q} \delta(\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q})\, M\, M \int dt\, G\, Q\, Q \right]$$
$$= \left[K^d\, K^d\, K^d\, K^{-d} \quad K\, K\, (K^{-1}U^{-1}) (U^b \epsilon^c k^a)^2 \right]$$
$$= \left[K^{2d+1+2a}\, U^{2b-1}\, \epsilon^{2c} \right] \tag{24}$$

$$a = -d - \frac{1}{2}, \quad b = \frac{1}{2}, \quad c = \frac{1}{2} \tag{25}$$

$$Q_k(t,t) \propto U^{1/2} \epsilon^{1/2} k^{-d-\frac{1}{2}} \tag{26}$$

Energy Spectrum

$$\begin{aligned} E(k,t) &:= \frac{1}{2} \int_{\underset{\sim}{k}'} \delta(|\underset{\sim}{k}'| - k)\, Q^{\alpha\alpha}_{\underset{\sim}{k}}(t,t) \\ &= \frac{1}{2} \int_{\underset{\sim}{k}'} \delta(|\underset{\sim}{k}'| - k)\, Q_k(t,t) \\ &= \frac{1}{2} A_d k^{d-1} Q_k(t,t) \end{aligned} \tag{27}$$

A_d : surface area of d-dim. unit sphere.

$$E(k) \propto U^{1/2} \epsilon^{1/2} k^{-3/2} \tag{28}$$

Inconsistent with the Kolmogorov spectrum

$$E(k) \propto \epsilon^{2/3} k^{-5/3} \tag{29}$$

This is due to including the sweeping effect which should not be included in the estimate of the energy transfer.

§ 4 Lagrangian renormalized approximation

Lagrangian variables

$X(\underset{\sim}{x}, s \mid t)$

s : labeling time , t : measuring Time

The quantity of X at time t of the fluid element which was at position $\underset{\sim}{x}$ at time s.

position function

$$\psi(\underset{\sim}{x}' t \,;\, \underset{\sim}{x}, s) := \delta(\underset{\sim}{x}' - y(\underset{\sim}{x}, s \mid t)) \quad \text{— (1)}$$

$y(\underset{\sim}{x}, s \mid t)$: position at time t of the fluid element which was at position $\underset{\sim}{x}$ at time s.

$$\frac{\partial}{\partial t}\psi(\underset{\sim}{x}', t \,;\, x, s) = -u^{\alpha}(\underset{\sim}{x}', t)\frac{\partial}{\partial x'^{\alpha}}\psi(\underset{\sim}{x}', t \,;\, \underset{\sim}{x}, s) \quad \text{— (2)}$$

$$X(\underset{\sim}{x}, s \mid t) = \int d\underset{\sim}{x}' \, X(\underset{\sim}{x}', t)\, \psi(\underset{\sim}{x}', t \,;\, \underset{\sim}{x}, s) \quad \text{— (3)}$$

Two-point Lagrangian correlation function $Q^{(L)}$

Two-point Lagrangian response function $G^{(L)}$

$$Q^{(L)\alpha\beta}(\underset{\sim}{x},t;\underset{\sim}{x}',t') := \begin{cases} \langle [Pu]^{\alpha}(\underset{\sim}{x},t'|t)\, u^{\beta}(\underset{\sim}{x}',t') \rangle & (t \geq t') \\ \langle u^{\alpha}(\underset{\sim}{x},t)\, [Pu]^{\beta}(x',t|t') \rangle & (t < t') \end{cases} \quad \text{—(4)}$$

$$G^{(L)\alpha\beta}(\underset{\sim}{x},t;\underset{\sim}{x}',t') := \left\langle \frac{\delta [Pu]^{\alpha}(\underset{\sim}{x}',t'|t)}{\delta f^{\beta}(\underset{\sim}{x}',t')} \right\rangle \quad \text{— (5)}$$

P : projection onto solenoidal component

In Fourier space

$$Q^{\alpha\beta\,(L)}_{\underset{\sim}{k}}(t,t') := \frac{1}{(2\pi)^3}\int d(\underset{\sim}{x}-\underset{\sim}{x}')\, e^{-i\underset{\sim}{k}(\underset{\sim}{x}-\underset{\sim}{x}')}\, Q^{\alpha\beta\,(L)}(\underset{\sim}{x},t;\underset{\sim}{x}',t')$$

$$= \begin{cases} \dfrac{1}{\delta(\underset{\sim}{0})}\left\langle D^{\alpha\gamma}_{\underset{\sim}{k}}\, u^{\gamma}_{\underset{\sim}{k}}(t'|t)\, u^{\beta}_{-\underset{\sim}{k}}(t')\right\rangle & (t \geq t') \\ \\ \dfrac{1}{\delta(\underset{\sim}{0})}\left\langle u^{\alpha}_{\underset{\sim}{k}}(t)\, D^{\beta\gamma}_{\underset{\sim}{k}}\, u^{\gamma}_{-\underset{\sim}{k}}(t|t')\right\rangle & (t < t') \end{cases} \qquad -(6)$$

$$G^{\alpha\beta\,(L)}_{\underset{\sim}{k}}(t,t') := \int d(\underset{\sim}{x}-\underset{\sim}{x}')\, e^{-i\underset{\sim}{k}(\underset{\sim}{x}-\underset{\sim}{x}')}\, G^{\alpha\beta\,(L)}(\underset{\sim}{x},t;\underset{\sim}{x}',t')$$

$$= \frac{1}{\delta(\underset{\sim}{0})}\left\langle \widetilde{G}^{\alpha\beta\,(L)}_{\underset{\sim}{k}\underset{\sim}{k}}(t,t')\right\rangle \qquad (t \geq t') \qquad ---(7)$$

$$\widetilde{G}^{\alpha\beta\,(L)}_{\underset{\sim}{k},\underset{\sim}{k}'}(t,t') = \frac{\delta\, D^{\alpha\gamma}_{\underset{\sim}{k}}\, u^{\gamma}_{\underset{\sim}{k}}(t)}{\delta f^{\beta}_{\underset{\sim}{k}'}(t')} \qquad (8)$$

$$\psi_{\underset{\sim}{k},\underset{\sim}{k}'}(t,t') := \frac{1}{(2\pi)^3}\int d\underset{\sim}{x}\int d\underset{\sim}{x}'\, e^{i\underset{\sim}{k}\underset{\sim}{x}'}\, e^{i\underset{\sim}{k}'\underset{\sim}{x}}\, \psi(\underset{\sim}{x};t;\underset{\sim}{x},t') \qquad (9)$$

$$\psi_{k}(t,t') = \frac{1}{\delta(\underset{\sim}{0})}\left\langle \psi_{\underset{\sim}{k},-\underset{\sim}{k}}(t,t')\right\rangle \qquad (10)$$

$$\frac{\partial}{\partial t} Q_{\underset{\sim}{k}}^{\alpha\beta}(t,t) = \int_{\underset{\sim}{p},\underset{\sim}{q}} \delta(\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q}) \left[H^{\alpha\beta}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t) + H^{\beta\alpha}_{-\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q}}(t) \right]$$

$$- 2\nu k^2 Q_{\underset{\sim}{k}}^{\alpha\beta}(t,t) \tag{10}$$

$$\frac{\partial}{\partial t} Q_{\underset{\sim}{k}}^{\alpha\beta\,(L)}(t,s) = \int_{\underset{\sim}{p},\underset{\sim}{q}} \delta(\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q})\, I^{\alpha\beta\,(L)}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t,s)$$

$$- \nu K_{\underset{\sim}{k}}^{\alpha\beta}(t,s) \tag{11}$$

$$\frac{\partial}{\partial t} G_{\underset{\sim}{k}}^{\alpha\beta\,(L)}(t,s) = \int_{\underset{\sim}{p},\underset{\sim}{q}} \delta(\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q})\, J^{\alpha\beta\,(L)}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t,s)$$

$$- \nu L_{\underset{\sim}{k}}^{\alpha\beta}(t,s) \tag{12}$$

Primitive expansion

$$Q_{\underset{\sim}{k}}^{\alpha\beta\,(L)}(t,s) = Q_{k}^{\alpha\beta\,(0)}(t,s) + O(\lambda) \tag{13}$$

$$G_{\underset{\sim}{k}}^{\alpha\beta\,(L)}(t,s) = Q_{k}^{\alpha\beta\,(0)}(t,s) + O(\lambda) \tag{14}$$

$$\psi_{\underset{\sim}{k}}^{\alpha\beta}(t,s) = \psi_{\underset{\sim}{k}}^{\alpha\beta}(t,s) + O(\lambda) \tag{15}$$

$$H^{\alpha\beta}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t) = \lambda^2 H^{\alpha\beta\,(2)}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t)\left[Q^{(0)}, G^{(0)}\right] + O(\lambda^3) \tag{16}$$

$$I^{\alpha\beta\,(L)}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t) = \lambda^2 I^{\alpha\beta\,(2)}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t)\left[Q^{(0)}, G^{(0)}, \psi^{(0)}\right] + O(\lambda^3) \tag{17}$$

$$J^{\alpha\beta\,(L)}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t) = \lambda^2 J^{\alpha\beta\,(3)}_{\underset{\sim}{k}\underset{\sim}{p}\underset{\sim}{q}}(t)\left[Q^{(0)}, G^{(0)}, \psi^{(0)}\right] + O(\lambda^3) \tag{18}$$

Inverse expansion

$$Q^{\alpha\beta\,(0)}_{\underset{\sim}{k}}(t,s) = Q^{\alpha\beta}_{\underset{\sim}{k}}(t,s)^{(L)} + O(\lambda) \quad (19)$$

$$G^{\alpha\beta\,(0)}_{\underset{\sim}{k}}(t,s) = G^{\alpha\beta}_{\underset{\sim}{k}}(t,s)^{(L)} + O(\lambda) \quad (20)$$

$$\left(\psi^{(0)}_{\underset{\sim}{k}}(t,s) = \psi_{\underset{\sim}{k}}(t,s) + O(\lambda) \quad (21) \right)$$

Renormalized expansion

$$H^{\alpha\beta}_{\underset{\sim}{kpq}}(t) = \lambda^2 H^{\alpha\beta}_{\underset{\sim}{kpq}}(t)\,[Q^{(L)}, G^{(L)}] + O(\lambda^3) \quad (22)$$

$$I^{\alpha\beta\,(L)}_{\underset{\sim}{kpq}}(t) = \lambda^2 I^{\alpha\beta}_{\underset{\sim}{kpq}}(t,s)\,[Q^{(L)}, G^{(L)}, \psi^{(0)}] + O(\lambda^3) \quad (23)$$

$$J^{\alpha\beta}_{\underset{\sim}{kpq}}(t) = \lambda^2 J^{\alpha\beta}_{\underset{\sim}{kpq}}(t,s)\,[Q^{(L)}, G^{(L)}, \psi^{(0)}] + O(\lambda^3) \quad (24)$$

Truncation

Retain the non-trivial leading order terms

in (22) (23) (24).

(10) (11) (12) $\longleftarrow$ (22) (23) (24)

$\Downarrow$

LRA equation.

§5 Properties of LRA

✓ Energy conservation ($\nu = 0$)

✓ When $\nu = 0$, compatible with
equipartion (3.1) and
FD relation (3.2).

· Realizability → not shown.

✓ Enegy spectrum

$$E(k) = K_0 \epsilon^{2/3} k^{-5/3} \quad (1)$$

$$K_0 \fallingdotseq 1.72 \quad (2)$$

Kolmogorov spectrum.

This is because

$\int_{\underset{\sim\sim}{p,q}}$ integrals in (12)

$$\begin{Bmatrix} \text{converge} \\ \text{diverge} \end{Bmatrix} \quad \text{for} \quad \begin{Bmatrix} a > -d-2 \\ a < -d-2 \end{Bmatrix} \quad (3)$$

as $q \to 0$, when $Q_k(t, \tau) \propto k^a$

Similarity form in the inertial range

$$G_k^{(L)}(t,s) = g\left(\epsilon^{e} k^{c}(t-s)\right) \qquad (4)$$

$$Q_k^{(L)}(t,s) = G_k^{(L)}(t,s)\, Q_k(s,s) \qquad (5)$$

$$Q_k(t,t) \propto \epsilon^{b} k^{a} \qquad (6)$$

$$[\epsilon] = [\Pi(K)] = \left[\int_{k} \int_{|k|>K} \int_{p,q} \delta(k-p-q)\, M M \int dt\, G Q Q\right]$$

$$= \left[K^{d} K^{d} K^{d} K^{-d} K K \left(\epsilon^{-e} K^{-c}\right)\left(\epsilon^{b} K^{a}\right)^{2}\right]$$

$$= \left[K^{2d+2-c+2a}\, \epsilon^{-e+2b}\right] \qquad — (7)$$

$$2d+2-c+2a = 0 \qquad (8)$$

$$-e+2b = 1 \qquad (9)$$

$$[\epsilon] = \left[\frac{L^2}{T^3}\right] \qquad [k] = \left[\frac{1}{L}\right]$$

$$\left[\frac{1}{T}\right] = \left[\epsilon^{1/3} k^{2/3}\right]$$

$$e = 1/3 \qquad c = 2/3 \qquad (10)$$

$$a = -d - \frac{2}{3}, \qquad b = \frac{2}{3} \qquad (11)$$

$$Q_k(t,t) \propto \epsilon^{2/3} k^{-d-2/3} \qquad (12)$$

$$E(k,t) \propto \epsilon^{2/3} k^{-5/3} \qquad (13)$$

§6 Wave turbulence

$$\frac{\partial}{\partial t} u^{\alpha}_{\underset{\sim}{k}} = \lambda \int_{\underset{\sim}{p},\underset{\sim}{q}} \delta(\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q})\, M^{\alpha\beta\gamma}_{\underset{\sim}{k}}\, u^{\beta}_{\underset{\sim}{p}}\, u^{\gamma}_{\underset{\sim}{q}}$$

$$- \nu k^2 u^{\alpha}_{\underset{\sim}{k}} + D^{\alpha\gamma}_{\underset{\sim}{k}} f^{\gamma}_{\underset{\sim}{k}}$$

$$+ \underline{\underline{L^{\alpha\beta}_{\underset{\sim}{k}}\, u^{\beta}_{\underset{\sim}{k}}}} \qquad (1)$$

Linearized equation

$$\frac{\partial}{\partial t} u^{\alpha}_{\underset{\sim}{k}} = L^{\alpha\beta}_{\underset{\sim}{k}}\, u^{\beta}_{\underset{\sim}{k}} \qquad (2)$$

$$u_{\underset{\sim}{k}}(t) \sim e^{i\omega_{\underset{\sim}{k}} t}\, u_{\underset{\sim}{k}} \qquad (3)$$

↑

Wave.

When L is large compared to the nonlinear term, λ - expansion can be justified

$\nu = 0$

$$G^{(0)}_{\underset{\sim}{k}}(t,t') \sim e^{i\omega_{\underset{\sim}{k}}(t-t')} \tag{4}$$

$$Q_{\underset{\sim}{k}}(t,t') \sim G^{(0)}_{\underset{\sim}{k}}(t,t')\, Q_{\underset{\sim}{k}}(t',t') \tag{5}$$

$$\frac{\partial}{\partial t} Q_k(t,t) = \int_{\underset{\sim}{p},\underset{\sim}{q}} \delta(\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q}) \int_{t_0}^{t} dt'\, M M\, G^{(0)}(t,t')\, Q(t,t')\, Q(t,t')$$

$$= \int_{\underset{\sim}{p},\underset{\sim}{q}} \delta(\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q}) \int_{t_0}^{t} dt'\, M M\, G^{(0)}(t,t')\, G^{(0)}(t,t')\, G^{(0)}(t,t')$$

$$\times \underbrace{Q(t',t')\, Q(t',t')}_{\downarrow} \tag{6}$$

$$Q(t,t)\quad Q(t,t)$$

slowly varying

$$Q_k = [\, \epsilon^{b} \omega_k^{c} k^{a} \,] \tag{7}$$

$$[\epsilon] = [\,\Pi(K)\,] = \left[\int_{\underset{\sim}{k},\ |\underset{\sim}{k}|>K} \int_{\underset{\sim}{p},\underset{\sim}{q}} \delta(\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q})\, M M \int dt\, G^{(0)}\, Q\, Q \right]$$

$$= [\, K^d K^d K^d K^{-d} K^2\, \omega_k^{-1} (\epsilon^{b} \omega_k^{c} k^{a})^2 \,] \tag{8}$$

$$\begin{cases} 2d + 2 + 2a = 0 \\ 2b = 1 \\ -1 + 2c = 0 \end{cases} \tag{9}$$

$$a = -d-1, \quad b = \frac{1}{2}, \quad c = \frac{1}{2} \tag{10}$$

$$Q_k \propto \epsilon^{1/2} \omega_k^{1/2} k^{-d-1} \qquad (11)$$

$$E(k) \propto k^{d-1} Q_k \propto \epsilon^{1/2} \omega_k^{1/2} k^{-2} \qquad (12)$$