Anisotropic spectrum of homogeneous turbulent shear flow in a Lagrangian renormalized approximation

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An analytical study of the anisotropic velocity correlation spectrum tensor in the inertial subrange of homogeneous turbulent shear flow is performed using a Lagrangian renormalized spectral closure approximation. The analysis shows that the spectrum in the asymptotic limit of infinitely large Reynolds numbers Re is determined by two nondimensional universal constants; theoretical estimates for the constants are provided. The anisotropic component of the spectrum at finite Re is more sensitive to large-scale turbulence structures than the isotropic component. A preliminary analysis of the effect of finite Re or the width of the inertial subrange is in qualitative agreement with direct numerical simulations. © *2003 American Institute of Physics.*
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I. INTRODUCTION

According to the Kolmogorov hypothesis,[1] turbulence statistics far from flow boundaries are locally homogeneous and isotropic at sufficiently large Reynolds numbers Re and at scales sufficiently smaller than the characteristic length scale L of the energy containing eddies. This hypothesis is widely accepted in literature. However, both Re and L must be finite, and the energy-containing eddies must be anisotropic in real flows, so that turbulence cannot be isotropic in a strict sense. Little is known about the degree of anisotropy in small-scale statistics. In this regard, recent experimental and numerical studies (see, for example, Refs. 2 and 3) suggest that the anisotropy may be substantial. The anisotropy appears not only at higher order moments, but also at second-order moments, which are the main concern of practical turbulence modeling. The study of anisotropy is therefore not only of theoretical but also of practical interest.

Recently, the authors[4] (hereafter referred to as IYK) derived a form of the velocity correlation spectrum tensor for small scales in homogeneous turbulent shear flow using a simple perturbation analysis. The anisotropic part of the tensor was determined by the rate of strain tensor S_{ij} of the mean flow, the mean energy dissipation rate ϵ per unit mass, the wave vector $\mathbf{k}$, and two nondimensional constants A_1 and A_2 (denoted by A and B, respectively in IYK). The scaling ($\propto k^{-13/3}$) of the anisotropic part of the velocity correlation spectrum is consistent with previous studies based on dimensional analysis including the one by Lumley[5] and the experiments by Wyngaard and Cote[6] (hereafter referred to as WC) and those by Saddoughi and Veeravalli[7] (hereafter referred to as SV). The form of the tensor was verified, and the two constants were estimated from direct numerical simulation (DNS) data. Tsuji[8] also obtained estimates of the two constants in the wall boundary layers of wind tunnel turbulence; their values are in good agreement with those reported by IYK.

It is a challenging problem to theoretically derive the anisotropic spectrum of turbulent shear flow. In this paper, we attempt to do so using a spectral closure approximation. Although extensive closure approximations have been proposed, there are very few that do not contain any *ad hoc* adjusting parameters, and are consistent not only with the Kolmogorov energy spectrum $E(k)=K_o\epsilon^{2/3}k^{-5/3}$ of homogeneous and isotropic turbulence, where K_o is the Kolmogorov constant, but also with the $k^{-2/3}$ scaling of the characteristic time in the inertial subrange of fully developed turbulence. To the authors' knowledge, there are only three closure approximations that have these properties and whose estimates of K_o are in fairly good agreement with the results from experiments and DNS. They are: the abridged Lagrangian history direct interaction approximation (ALHDIA),[9] the strain-based abridged Lagrangian history direct interaction approximation (SBALHDIA),[10] and the Lagrange renormalized approximation (LRA)[11] (hereafter referred to as K81). All of these are Lagrangian spectral closures.

It seems worthwhile to make here a few remarks. The first concerns the so-called intermittency corrections to the forms of the velocity correlation spectra. If the energy spectrum $E(k)$ is to be modified to $E(k)\propto k^{-5/3-\mu}$ with $\mu\neq 0$, the Kolmogorov constant K_o loses its meaning. Similarly, if the scaling $\propto k^{-13/3}$ of the anisotropic part of the velocity correlation spectrum is to be modified, the constants A_1 and A_2 lose their meaning. The present status of our understanding on the inertial subrange spectrum $E(k)$ seems to be well summarized by the statement "There is a general belief (although contested often enough) that the spectral exponent gets slightly modified by small-scale intermittency. This modification, even if exists, is small and cannot be accommodated in a consistent and satisfactory way given other uncertainties in the data" by Sreenivasan.[12] This also seems to be the case regarding the anisotropic part of the velocity

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correlation; although there have been pioneering studies by experiments[13,14] and DNS[3,15] on the scaling of the anisotropic components based on SO(3) decomposition, the uncertainties do not seem small enough to fix the modification, if it exists.

Although the uncertainties may be better controlled in DNS than in experiments, it is still difficult to fix by DNS the modification in the limit of large Reynolds number, because of the limitation of the attainable resolution or the Reynolds number, which implies the limitation of the width of the realized inertial subrange. (Recent high resolution DNSs[16] with the number of grid points up to 4096^3 suggest $\mu \sim 0.10$, where the highest Taylor microscale Reynolds number R_λ achieved in the DNSs is 1201. This value of μ is a little larger than the value $-0.03 \lesssim \mu \lesssim 0.03$ suggested from measurements[17] in the atmospheric surface layer turbulences over the range of $R_\lambda = 2800-12\,700$.)

The second remark concerns the capability of spectral closures to describe intermittency effects. Spectral closures have so far made contributions to the understanding of turbulence such as quantitative predictions of energy spectra, the derivation of the eddy viscosity based on the Navier–Stokes equations, etc., but also have shortcomings including the one associated with the intermittency of small scales (see, for example, the review by Kraichnan[18]). Regarding high-order velocity moments, it is not surprising that spectral closures such as the ALHDIA and LRA do not capture intermittency effects on them, because the closures concern only the second-order velocity moments. Regarding the second-order moments, little progress has been made so far by spectral closures on the intermittency. One might therefore think that they are incapable of predicting anomalous scaling. It is however to be recalled that exact closure equations for simple models such as randomly advected passive scalar and vectors with or without pressure have solutions, the so-called zero modes, that exhibit anomalous scaling (see, for example, Refs. 19–26). The above spectral closure equations for turbulence obeying the Navier–Stokes equations are similar in a sense to the exact closure equations for these models which have solutions exhibiting anomalous scaling [see the discussion after Eqs. (50)–(52)]. It is therefore difficult to exclude at present the possibility that spectral closure equations may yield anomalous scaling. However, the so-called zero-mode analysis of the closure equations is not easy, and is outside the scope of the present paper.

The above-mentioned spectral closures for second-order moments are obtained by truncating certain renormalized perturbation (RP) series at the lowest nontrivial order. It would be of theoretical interest to know the consequence of continuing the RP series to higher orders, or applying the RP approach to higher order moments. However only few studies have made of the consequence (but see Refs. 27–30). The present authors think that the issue remains disputable, and to be studied further, but it is again outside the scope of the present paper.

In this paper, we assume that the intermittency correction, if it exists, is small, in accordance with Sreenivasan,[12] and that Lagrangian spectral closures may be applicable to obtain an approximation of the energy spectrum for isotropic turbulence. The latter assumption is supported in part by the good quantitative agreement of the spectrum by LRA for isotropic turbulence at large Re with experiments as shown in Fig. 1 of Ref. 31. It is therefore tempting to apply closures also to anisotropic turbulences. However, there have been only few studies on anisotropic turbulences, in contrast to isotropic turbulence, on the basis of spectral closures. (They include studies on axisymmetric turbulence by Herring,[32] and on turbulent shear flow by Leslie,[33] Cambon *et al.*,[34] Bertoglio,[35] Rubinstein *et al.*,[36] and Yoshizawa.[37]) In particular, to our knowledge, no analytical studies have been performed on anisotropic turbulence using any of the above-noted Lagrangian closures. This is presumably because of the complexity of the equations that must be analyzed. Fortunately, the recent study by IYK suggests that the analysis may be greatly simplified by properly taking into account the symmetry of the problem. These considerations encourage us to analyze the Lagrangian turbulence closure equations for homogeneous turbulent shear flow. We consider in the following the LRA because its equations are the simplest amongst the three closures. The analysis is based on a systematic perturbation method, and introduces no *ad hoc* parameters in the analysis of the asymptotic limit of infinitely large Re.

II. LRA EQUATIONS FOR HOMOGENEOUS TURBULENT SHEAR FLOW

In this paper, we consider an ensemble of the turbulent velocity fields $\mathbf{u}(\mathbf{x},t)$ for an incompressible fluid obeying the Navier–Stokes equations,

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x_l \partial x_l}\right) u_i(\mathbf{x},t) = -u_m(\mathbf{x},t)\frac{\partial u_i}{\partial x_m}(\mathbf{x},t) - \frac{\partial p}{\partial x_i}(\mathbf{x},t), \tag{1}$$

$$\frac{\partial u_i}{\partial x_i}(\mathbf{x},t) = 0, \tag{2}$$

where ν is the kinematic viscosity coefficient, the density is assumed to be unity, and p is the pressure. The summation convention is used for repeated indices. The statistical average taken over the ensemble for a quantity X is denoted by $\langle X \rangle$. Let us decompose the velocity field $\mathbf{u}$ into its mean and fluctuating terms as $\mathbf{u}(\mathbf{x},t) = \langle \mathbf{u}(\mathbf{x},t) \rangle + \tilde{\mathbf{u}}(\mathbf{x},t)$, where $\langle \mathbf{u}(\mathbf{x},t) \rangle$ is the mean flow and $\tilde{\mathbf{u}}(\mathbf{x},t)$ is the fluctuation. Then $\langle \mathbf{u} \rangle$ and $\tilde{\mathbf{u}}$ obey

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x_l \partial x_l}\right) \langle u_i(\mathbf{x},t) \rangle = -\langle u_m(\mathbf{x},t) \rangle \frac{\partial \langle u_i(\mathbf{x},t) \rangle}{\partial x_m} - \frac{\partial}{\partial x_m} \langle \tilde{u}_m(\mathbf{x},t) \tilde{u}_i(\mathbf{x},t) \rangle - \frac{\partial \langle p(\mathbf{x},t) \rangle}{\partial x_i}, \tag{3}$$

$$\frac{\partial \langle u_i(\mathbf{x},t) \rangle}{\partial x_i} = 0, \tag{4}$$

and

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial x_l\partial x_l}\right)\tilde{u}_i(\mathbf{x},t)$$
$$=-\tilde{u}_m(\mathbf{x},t)\frac{\partial\tilde{u}_i}{\partial x_m}(\mathbf{x},t)+\frac{\partial}{\partial x_m}\langle\tilde{u}_m(\mathbf{x},t)\tilde{u}_i(\mathbf{x},t)\rangle$$
$$-\frac{\partial\tilde{p}}{\partial x_i}(\mathbf{x},t)-\langle u_m(\mathbf{x},t)\rangle\frac{\partial\tilde{u}_i}{\partial x_m}(\mathbf{x},t)$$
$$-\frac{\partial\langle u_i(\mathbf{x},t)\rangle}{\partial x_m}\tilde{u}_m(\mathbf{x},t), \tag{5}$$

$$\frac{\partial\tilde{u}_i}{\partial x_i}(\mathbf{x},t)=0, \tag{6}$$

respectively, where $\tilde{p}=p-\langle p\rangle$. Equation (3) for the mean flow $\langle\mathbf{u}\rangle$ contains single-time second-order moments of $\tilde{\mathbf{u}}$, which are determined in principle by Eqs. (5) and (6).

In the derivation of the LRA equations, use of the so-called Lagrangian position function ψ plays a key role (see K81). The function is defined by

$$\psi(\mathbf{y},t;\mathbf{x},t')\equiv\delta[\mathbf{y}-\mathbf{a}(\mathbf{x},t';t)],$$

where $\mathbf{a}(\mathbf{x},t';t)$ is the position at time t of the fluid element, which was at $\mathbf{x}$ at time t'. In terms of ψ, the generalized velocity $\mathbf{v}(\mathbf{x},t';t)$ defined as the velocity fluctuation at time t of the fluid particle that was at position $\mathbf{x}$ at time t', is given by

$$\mathbf{v}(\mathbf{x},t';t)\equiv\int d\mathbf{y}\,\psi(\mathbf{y},t;\mathbf{x},t')\tilde{\mathbf{u}}(\mathbf{y},t). \tag{7}$$

The function ψ obeys

$$\frac{\partial}{\partial t}\psi(\mathbf{y},t;\mathbf{x},t')=-[\langle u_m(\mathbf{y},t)\rangle+\tilde{u}_m(\mathbf{y},t)]$$
$$\times\frac{\partial}{\partial y_m}\psi(\mathbf{y},t;\mathbf{x},t'). \tag{8}$$

Equation (7) implies that the evolution of $\mathbf{v}(\mathbf{x},t';t)$ with respect to time t is known from those of $\langle\mathbf{u}(\mathbf{x},t)\rangle$, $\tilde{\mathbf{u}}(\mathbf{x},t)$ and $\psi(\mathbf{y},t;\mathbf{x},t')$, which are given by Eqs. (3)–(6) and (8).

In general, the performance of an approximation may depend crucially on the choice of quantities (called ''representatives'' in K81) in terms of which the approximation is constructed. It is therefore important to choose proper representatives, as stressed in K81. The LRA uses the following representatives: the Lagrangian two-time and two-point velocity correlation Q and the Lagrangian response function G, defined as

$$Q_{ij}(\mathbf{x},t;\mathbf{x}',t')\equiv P_S^{(\mathbf{x})}\langle[v_i(\mathbf{x},t';t)]v_j(\mathbf{x}',t')\rangle, \tag{9}$$

$$\int d\mathbf{x}'\,G_{ij}(\mathbf{x},t;\mathbf{x}',t')\,\delta v_j(\mathbf{x}',t';t')$$
$$\equiv P_S^{(\mathbf{x})}\langle[\delta v_i(\mathbf{x},t';t)]\rangle, \tag{10}$$

where $\delta\mathbf{v}(\mathbf{x},t';t')$ is an infinitesimal disturbance field at time t' that is statistically independent of the disturbed velocity field, $\delta\mathbf{v}(\mathbf{x},t';t)$ is the response of $\mathbf{v}(\mathbf{x},t';t)$ at time $t\geqslant t'$, and $P_S^{(\mathbf{x})}$ is the operator that projects a vector field on $\mathbf{x}$ to its solenoidal component.

The LRA procedure proposed in K81 may be applied to the field obeying Eqs. (5) and (6). This yields a closed set of equations for $\langle\mathbf{u}\rangle$, Q, and G. Hereafter, these are referred to as the LRA equations. Although the LRA is applicable to turbulence in an arbitrary domain and mean flow in principle, we consider here the simplest but nontrivial case in which the flow domain is unbounded in each of the three Cartesian coordinate directions, and $\langle\mathbf{u}\rangle$ is given by a linear function of the position vector $\mathbf{x}$,

$$\langle u_m(\mathbf{x})\rangle=S_{mn}x_n, \tag{11}$$

where S_{mn} is a time independent tensor. The incompressibility and stationarity of $\langle\mathbf{u}\rangle$ requires that $S_{mm}=0$ and $S_{mn}S_{nl}=S_{ln}S_{nm}$, respectively. The LRA equations are then compatible with the homogeneity of Q and G, i.e., if $Q(\mathbf{x},t_0;\mathbf{x}',t_0)$ depends on $\mathbf{x}$ and $\mathbf{x}'$ only through $\mathbf{x}-\mathbf{x}'$ at an initial instant t_0, then it is also true for $Q(\mathbf{x},t;\mathbf{x}',t')$ and $G(\mathbf{x},t;\mathbf{x}',t')$ for $t\geqslant t'\geqslant t_0$. We assume here that Q and G are statistically homogeneous. It is then convenient to introduce the Fourier transforms of Q and G with respect to $\mathbf{x}-\mathbf{x}'$,

$$\hat{Q}_{ij}(\mathbf{k},t,t')=\frac{1}{(2\pi)^3}\int d^3(\mathbf{x}-\mathbf{x}')$$
$$\times Q_{ij}(\mathbf{x},t;\mathbf{x}',t')\mathrm{e}^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \tag{12}$$

$$\hat{G}_{ij}(\mathbf{k},t,t')=\int d^3(\mathbf{x}-\mathbf{x}')G_{ij}(\mathbf{x},t;\mathbf{x}',t')\mathrm{e}^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \tag{13}$$

respectively.[38] The tensor $\hat{Q}$ will be referred to as the velocity correlation spectrum tensor, and we will omit the hat $\hat{}$ for convenience.

The LRA equations in Fourier space representation are given as follows:

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q_{ij}(\mathbf{k},t,t)=D_{ij}(\mathbf{k},t)+K_{ijmn}(\mathbf{k},t)S_{mn}, \tag{14}$$

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)Q_{ij}(\mathbf{k},t,s)=J_{ij}^{Q}(\mathbf{k},t,s)+L_{ijmn}^{Q}(\mathbf{k},t,s)S_{mn}, \tag{15}$$

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)G_{ij}(\mathbf{k},t,s)=J_{ij}(\mathbf{k},t,s)+L_{ijmn}(\mathbf{k},t,s)S_{mn}, \tag{16}$$

$$G_{ij}(\mathbf{k},t,t)=P_{ij}(\mathbf{k}), \tag{17}$$

where

$$D_{ij}(\mathbf{k},t)=H_{ij}(\mathbf{k},t)+H_{ji}(-\mathbf{k},t), \tag{18}$$

$$H_{ij}(\mathbf{k},t)=\sum_{\mathbf{p},\mathbf{q}}^{\triangle}\int_{t_0}^{t}ds'[-P_{iab}(\mathbf{k})P_{cde}(\mathbf{p})G_{ac}(\mathbf{p},t,s')$$
$$\times Q_{bd}(\mathbf{q},t,s')Q_{je}(-\mathbf{k},t,s')+\tfrac{1}{2}P_{iab}(\mathbf{k})P_{ecd}(\mathbf{k})$$
$$\times G_{je}(-\mathbf{k},t,s')Q_{ac}(\mathbf{p},t,s')Q_{bd}(\mathbf{q},t,s')], \tag{19}$$

$$K_{ijmn}(\mathbf{k},t)=k_m\frac{\partial}{\partial k_n}Q_{ij}(\mathbf{k},t)-[P_{im}(\mathbf{k})-\hat{k}_i\hat{k}_m]Q_{nj}(\mathbf{k},t)$$
$$-[P_{jm}(\mathbf{k})-\hat{k}_j\hat{k}_m]Q_{ni}(-\mathbf{k},t), \tag{20}$$

$$J^Q_{ij}(\mathbf{k},t,s)=-\sum_{\mathbf{p},\mathbf{q}}^{\triangle}\int_s^t ds'\,P_{ib}(-\mathbf{k})k_a\frac{p_bp_cp_l}{p^2}$$
$$\times Q_{ca}(-\mathbf{q},t,s')Q_{lj}(\mathbf{k},t,s), \tag{21}$$

$$L^Q_{ijmn}(\mathbf{k},t,s)=-P_{im}(\mathbf{k})Q_{nj}(\mathbf{k},t,s), \tag{22}$$

$$J_{ij}(\mathbf{k},t,s)=-\sum_{\mathbf{p},\mathbf{q}}^{\triangle}\int_s^t ds'\,P_{ib}(-\mathbf{k})k_a\frac{p_bp_cp_l}{p^2}$$
$$\times Q_{ca}(-\mathbf{q},t,s')G_{lj}(\mathbf{k},t,s), \tag{23}$$

$$L_{ijmn}(\mathbf{k},t,s)=-P_{im}(\mathbf{k})G_{nj}(\mathbf{k},t,s), \tag{24}$$

$$k\equiv|\mathbf{k}|,\quad \hat{\mathbf{k}}\equiv\frac{\mathbf{k}}{k},\quad P_{ij}(\mathbf{k})\equiv\delta_{ij}-\hat{k}_i\hat{k}_j,$$

$$P_{imn}(\mathbf{k})\equiv k_mP_{in}(\mathbf{k})+k_nP_{im}(\mathbf{k}),$$

$$\sum_{\mathbf{p},\mathbf{q}}^{\triangle}\equiv\int d^3\mathbf{p}d^3\mathbf{q}\,\delta(\mathbf{k}-\mathbf{p}-\mathbf{q}),$$

t_0 is the initial time, and $t\geqslant s\geqslant t_0$. From Eqs. (15)–(17) and (21)–(24), we have

$$Q_{ij}(\mathbf{k},t,s)=G_{ia}(\mathbf{k},t,s)Q_{aj}(\mathbf{k},s,s). \tag{25}$$

The LRA equations, Eqs. (14)–(24), are therefore a closed set for $Q_{ij}(\mathbf{k},t)\equiv Q_{ij}(\mathbf{k},t,t)$ and $G_{ij}(\mathbf{k},t,s)$. Hereafter, whenever Q appears without time indices, it will denote the one-time correlation $Q_{ij}(\mathbf{k},t)$ and not the two-time correlation $Q_{ij}(\mathbf{k},t,s)$. The LRA equations, Eqs. (14)–(24), are compatible with the reflection symmetry of Q and G, i.e., $Q_{ij}(-\mathbf{k},t)=Q_{ij}(\mathbf{k},t)$ and $G_{ij}(-\mathbf{k},t,s)=G_{ij}(\mathbf{k},t,s)$ where the former is equivalent to

$$Q_{ji}(\mathbf{k},t)=Q_{ij}(\mathbf{k},t). \tag{26}$$

We assume the reflection symmetry of Q and G.

III. INERTIAL RANGE ANALYSIS OF THE LRA EQUATIONS

The inertial subrange solutions Q and G of the LRA equations, Eqs. (14)–(24), for $S_{mn}=0$ were obtained in Ref. 39. They are

$$Q_{ij}(\mathbf{k},t)=Q^{(0)}_{ij}(\mathbf{k};\epsilon)\equiv\frac{K_o}{4\pi}\epsilon^{2/3}k^{-11/3}P_{ij}(\mathbf{k}), \tag{27}$$

$$G_{ij}(\mathbf{k},t,s)=G^{(0)}_{ij}(\mathbf{k},\tau;\epsilon)\equiv G^{(0)}(\xi)P_{ij}(\mathbf{k}), \tag{28}$$

where $\xi\equiv\tau/T_L(k)$, $\tau\equiv t-s$, $T_L(k)\equiv\epsilon^{-1/3}k^{-2/3}$, K_o is the Kolmogorov constant, and $G^{(0)}$ is a universal function. The LRA gives

$$K_o=1.72. \tag{29}$$

The dependence of $G^{(0)}$ on the normalized time difference ξ is shown in Fig. 1. The function $G^{(0)}$ monotonically decays with ξ, and $G^{(0)}\to\exp(-c\xi)$ as $\xi\to\infty$, where c is a nondimensional constant of order unity.

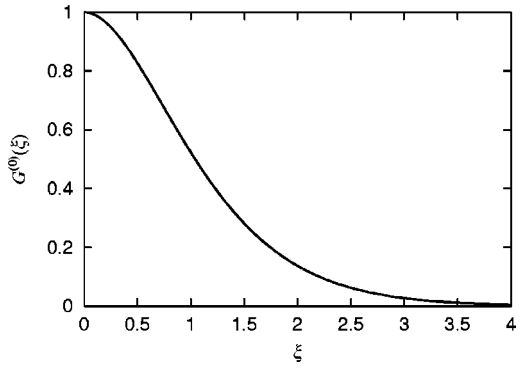

FIG. 1. Isotropic Lagrangian response function $G^{(0)}$ as a function of the normalized time difference $\xi=\tau/T_L(k)$.

In the inertial subrange of homogeneous turbulent shear flow, Q and G can be written as

$$Q_{ij}(\mathbf{k},t)=Q^{(0)}_{ij}(\mathbf{k};\epsilon(t))+Q^{(1)}_{ij}(\mathbf{k},t), \tag{30}$$

$$G_{ij}(\mathbf{k},t+\tau,t)=G^{(0)}_{ij}(\mathbf{k},\tau;\epsilon(t))+G^{(1)}_{ij}(\mathbf{k},t+\tau,t), \tag{31}$$

where $\epsilon(t)$ may depend on time t. In the present problem, there are at least three types of time scales that may be distinguished from each other: (i) the time scale T_S associated with the mean shear rate and given by $T_S\equiv1/S$ where $S\equiv\max_{ij}|S_{ij}|$, (ii) the time scale $T_t(k)$ characterizing the time dependence of the single-time correlation $Q_{ij}(\mathbf{k},t)$, and (iii) the time scale $T_\tau(k)$ characterizing the decay with respect to the time difference τ of the two-time correlations $G_{ij}(\mathbf{k},t+\tau,t)$ and $Q_{ij}(\mathbf{k},t+\tau,t)$.

In this paper, we assume that in the inertial subrange (not the entire wavenumber range):

(A-1) the corrections $Q^{(1)}$ and $G^{(1)}$ are small enough that we may discard terms second or higher order in $(Q^{(1)},G^{(1)})$ in the LRA equations;
(A-2) the mean shear rate S is small enough that the time scale $T_S=1/S$ is much larger than $T_\tau(k)$, i.e., $\delta(k)\equiv T_\tau(k)/T_S=ST_\tau(k)\ll1$;
(A-3) the time scale $T_t(k)$ is also much larger than $T_\tau(k)$, i.e., $\mu(k)\equiv T_\tau(k)/T_t(k)\ll1$;
(A-4) the response function $G_{ij}(\mathbf{k},t+\tau,t)$ is negligibly small for the time difference $\tau\gg T_\tau(k)$.

The consistency of these assumptions with the resulting Q and G will be discussed at the end of this section for (A-1)–(A-3) and in Sec. IV for (A-4).

For $|s'-s|\ll T_t(k)$, we have

$$Q_{ij}(\mathbf{k},s')\sim Q_{ij}(\mathbf{k},s)+(s'-s)\frac{\partial}{\partial s}Q_{ij}(\mathbf{k},s), \tag{32}$$

$$G_{ij}(\mathbf{k},t,s')=G_{ij}(\mathbf{k},s'+(t-s'),s')$$
$$\sim G_{ij}(\mathbf{k},s+(t-s'),s)$$
$$+(s'-s)\frac{\partial}{\partial s}G_{ij}(\mathbf{k},s+(t-s'),s), \tag{33}$$

from the definition of $T_t(k)$. Assumption (A-4) combined with (A-3) implies that the main contribution in the s' integrals of Eqs. (19) and (23) comes from $t-s' \ll T_t(k)$. Taking this into account, substituting Eqs. (30) and (31) into the LRA equations with Eq. (25), and discarding the second- or higher-order terms in $(Q^{(1)},G^{(1)})$ by virtue of assumption (A-1), we obtain the following closed set of equations for $Q^{(1)}$ and $G^{(1)}$:

$$\mathcal{D}_{ij}[Q^{(1)},G^{(1)}](\mathbf{k},t) = -K^{(0)}_{ijmn}(\mathbf{k},t)S_{mn} + \frac{\partial}{\partial t}Q^{(0)}_{ij}(\mathbf{k},t), \quad (34)$$

$$\mathcal{N}_{ij}[Q^{(1)},G^{(1)}](\mathbf{k},s+\tau,s)$$

$$\equiv \frac{\partial}{\partial \tau}G^{(1)}_{ij}(\mathbf{k},s+\tau,s) - \mathcal{J}_{ij}[Q^{(1)},G^{(1)}](\mathbf{k},s+\tau,s)$$

$$= L^{(0)}_{ijmn}(\mathbf{k},s+\tau,s)S_{mn} + \tilde{J}_{ij}(\mathbf{k},s+\tau,s), \quad (35)$$

$$G^{(1)}(\mathbf{k},s,s) = 0, \quad (36)$$

where we have discarded terms containing $\delta(k)Q^{(1)}$, $\delta(k)G^{(1)}$, $\mu(k)Q^{(1)}$, or $\mu(k)G^{(1)}$, because of assumptions (A-2) and (A-3). The functionals $\mathcal{D}$, $\mathcal{J}$, and $\mathcal{N}$ are linear in $(Q^{(1)},G^{(1)})$. The expressions of $\mathcal{D}$, $\mathcal{J}$, and $\tilde{J}$ are rather lengthy, and are given in Appendix A. The symbols $K^{(0)}$ and $L^{(0)}$ are the same as K in Eq. (20) and L in Eq. (24), respectively, but with Q and G replaced by $Q^{(0)}$ and $G^{(0)}$.

Equations (34)–(36) are satisfied by

$$Q^{(1)}(\mathbf{k},t) = Q^S(\mathbf{k},t) + Q^T(\mathbf{k},t), \quad (37)$$

$$G^{(1)}(\mathbf{k},t+\tau,t) = G^S(\mathbf{k},t+\tau,t) + G^T(\mathbf{k},t+\tau,t), \quad (38)$$

where (Q^S,G^S) and (Q^T,G^T) satisfy

$$\mathcal{D}_{ij}[Q^S,G^S](\mathbf{k},t) = -K^{(0)}_{ijmn}(\mathbf{k},t)S_{mn}, \quad (39)$$

$$\mathcal{N}_{ij}[Q^S,G^S](\mathbf{k},t+\tau,t) = L^{(0)}_{ijmn}(\mathbf{k},t+\tau,t)S_{mn}, \quad (40)$$

and

$$\mathcal{D}_{ij}[Q^T,G^T](\mathbf{k},t) = \frac{\partial}{\partial t}Q^{(0)}_{ij}[\mathbf{k},\epsilon(t)], \quad (41)$$

$$\mathcal{N}_{ij}[Q^T,G^T](\mathbf{k},t+\tau,t) = \tilde{J}_{ij}(\mathbf{k},t+\tau,t), \quad (42)$$

respectively.

By considering the isotropy of $Q^{(0)}$ and $G^{(0)}$ as well as the involved operators, one can show that Eqs. (39) and (40) are satisfied by

$$Q^S_{ij}(\mathbf{k},t) = X_{ijmn}(\mathbf{k},t)S_{mn}, \quad (43)$$

$$G^S_{ij}(\mathbf{k},t+\tau,t) = Y_{ijmn}(\mathbf{k},t+\tau,t)S_{mn}, \quad (44)$$

where X and Y are isotropic fourth-order tensors, and may be written without loss of generality in the form

$$X_{ijmn}(\mathbf{k},t) = a_1(k,t)[P_{im}(\mathbf{k})P_{jn}(\mathbf{k}) + P_{in}(\mathbf{k})P_{jm}(\mathbf{k})] + a_2(k,t)P_{ij}(\mathbf{k})\hat{k}_m\hat{k}_n, \quad (45)$$

$$Y_{ijmn}(\mathbf{k},t+\tau,t) = b_1(k,t+\tau,t)P_{im}(\mathbf{k})P_{jn}(\mathbf{k}) + b_2(k,t+\tau,t)P_{in}(\mathbf{k})P_{jm}(\mathbf{k}) + b_3(k,t+\tau,t)P_{ij}(\mathbf{k})\hat{k}_m\hat{k}_n. \quad (46)$$

Here we have used Eq. (26) to derive Eq. (45).

We will attempt to find a solution for $a_\alpha(k,t)$ and $b_\beta(k,t)$ in the following similarity forms:

$$a_\alpha(k,t) = A_\alpha \epsilon^{\chi_1} k^{\chi_2},$$

$$b_\beta(k,t+\tau,t) = \epsilon^{\chi_3} k^{\chi_4} B_\beta(\epsilon^{\chi_5} k^{\chi_6}\tau),$$

where $\epsilon = \epsilon(t)$, and A_α and $B_\beta(\cdot)$ are constants and functions, respectively. Here, the Greek indices α and β represent $\{1,2\}$ and $\{1,2,3\}$, respectively. By substituting the similarity forms of Q^S and G^S into Eqs. (39) and (40), we can verify that the $\mathbf{p}$, $\mathbf{q}$ integrals in $\mathcal{D}$ and $\mathcal{N}$ converge in the limit of both large and small wavenumbers, and the terms on the left- and right-hand sides of each equation have the same scaling if and only if

$$\chi_1 = 1/3, \quad \chi_2 = -13/3, \quad \chi_3 = -1/3,$$

$$\chi_4 = -2/3, \quad \chi_5 = 1/3, \quad \chi_6 = 2/3.$$

Therefore, we have

$$a_\alpha(k,t) = A_\alpha \epsilon^{1/3} k^{-13/3}, \quad b_\beta(k,t+\tau,t) = B_\beta(\xi)T_L(k), \quad (47)$$

in the inertial subrange, where $T_L(k) \equiv \epsilon^{-1/3}k^{-2/3}$ and $\xi \equiv \tau/T_L(k)$.

Similarly, it can be shown that Eqs. (41) and (42) are satisfied by (Q^T,G^T) in the following form:

$$Q^T_{ij}(\mathbf{k},t) = A^T \epsilon^{1/3} k^{-13/3} T_\epsilon^{-1} P_{ij}(\mathbf{k}), \quad (48)$$

$$G^T_{ij}(\mathbf{k},t+\tau,t) = B^T(\xi)T_L(k)T_\epsilon^{-1}P_{ij}(\mathbf{k}), \quad (49)$$

where $T_\epsilon \equiv \epsilon/(d\epsilon/dt)$, and A^T and $B^T(\cdot)$ are a constant and function, respectively. A comparison of Eq. (48) and Eq. (43) with Eqs. (45) and (47) suggests that $Q^T/Q^S = O(T_S/T_\epsilon)$. In this paper, we consider quasi-stationary turbulence, in the sense that T_ϵ is much larger than T_S, so that $Q^T/Q^S = O(T_S/T_\epsilon) \ll 1$. We may therefore neglect Q^T. Furthermore, Q^T and G^T are isotropic tensors and do not contribute to the anisotropic part of Q and G, which is the subject of this paper.

If there exists a homogeneous solution (Q^H,G^H) satisfying Eqs. (34)–(36), i.e.,

$$\mathcal{D}_{ij}[Q^H,G^H](\mathbf{k},t) = 0, \quad (50)$$

$$\mathcal{N}_{ij}[Q^H,G^H](\mathbf{k},t,s) = 0, \quad (51)$$

$$G^H(\mathbf{k},s,s) = 0, \quad (52)$$

then the right-hand sides of Eqs. (37) and (38) with Q^H and G^H added, respectively, also satisfy Eqs. (34)–(36). Equations similar to Eqs. (50)–(52) have been known, but presumably because of its difficulty, the analysis (the so-called zero-mode analysis) has been mostly limited to equations for simple models including randomly advected passive scalar, and vectors with or without pressure.[19–26] Recently, the zero-mode analyses of equations derived by certain linearization of turbulence closures were performed by Yoshida and Kaneda[26] and L'vov *et al.*[40] The equations analyzed by them are different from each other, and discard the correction $G^{(1)}$ to the response function, but their structure is similar to that of Eq. (50). These studies suggest that the homogeneous so-

lutions (zero modes) may yield anomalous scalings that cannot be derived by simple dimensional argument. We cannot at present exclude the possibility of the existence of the zero-mode (Q^H, G^H) which may affect $Q^{(1)}$. However, the facts that (i) the scaling $k^{-13/3}$ in the inertial subrange given by Q^S in Eq. (43) with Eqs. (45) and (47) is in good agreement with DNS in IYK and experiments by WC, SV, and Tsuji[8] and (ii) the tensor form Q^S is also in good agreement with the DNS, suggest that the possible effect of the zero mode in the present problem is not very significant.

In the following, we confine ourselves to the analysis of Q^S and G^S. The tensor form and scaling of Q^S in Eq. (43) with Eqs. (45) and (47) are the same as those obtained in IYK. [The constants A and B in IYK are equivalent to A_1 and A_2 in Eq. (47).] The $k^{-13/3}$ dependence of Q^S is in agreement with previous studies based on dimensional analysis, including that by Lumley.[5]

In concluding this section, let us consider the consistency of assumptions (A-1)–(A-3) with the resulting solution $(Q^{(1)}, G^{(1)}) = (Q^S, G^S)$. For $(Q^{(1)}, G^{(1)}) = (Q^S, G^S)$, we may redefine $T_\tau(k)$ and $T_t(k)$ as $T_\tau(k) \equiv T_L(k)$ and $T_t(k) \equiv T_\epsilon$, respectively, so that $\delta(k) = T_L(k)/T_S = S\epsilon^{-1/3}k^{-2/3}$ and $\mu(k) = T_L(k)/T_\epsilon = (d\epsilon/dt)\epsilon^{-4/3}k^{-2/3}$. Consequently, $Q^{(1)}/Q^{(0)} = O(\delta(k))$, $G^{(1)}/G^{(0)} = O(\delta(k))$. This suggests that at sufficiently large wavenumbers k such that $\delta(k) \ll 1$ and $\mu(k) \ll 1$ in the inertial subrange of turbulence at sufficiently large Re, assumptions (A-1)–(A-3) are well satisfied.

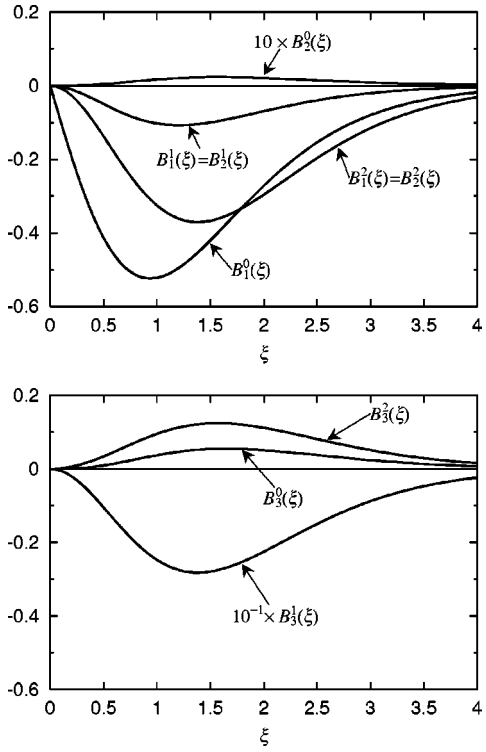

FIG. 2. Universal functions $B_\beta^\alpha(\xi)$ that define the anisotropic correction $G^{(1)}$ of the Lagrangian response function as functions of the nondimensionalized time difference $\xi = \tau/T_L(k)$. Values of $B_2^0(\xi)$ and $B_3^1(\xi)$ are rescaled.

IV. ESTIMATE OF THE UNIVERSAL CONSTANTS

The constants A_α and functions $B_\beta(\xi)$ that determine $Q^{(1)}$ and $G^{(1)}$, respectively, can be estimated from the LRA equations. For this purpose, it is convenient to introduce normalized functions, defined by

$$\overline{B}_\beta(\xi) \equiv B_\beta(\xi)/G^{(0)}(\xi).$$

From Eq. (40) with Eqs. (43)–(47), we obtain a closed set of integral differential equations for the functions $\overline{B}_\beta$, after performing some algebra, in the following form:

$$\frac{d^2}{d\xi^2}\overline{B}_\beta(\xi) = \int\int_\triangle dp\, dq\, U_{\beta\gamma}(p,q,\xi)\overline{B}_\gamma(q^{2/3}\xi) + A_1 V_\beta(\xi) + A_2 W_\beta(\xi), \tag{53}$$

$$\overline{B}_\beta(\xi) = 0, \quad \left.\frac{d\overline{B}_\beta(\xi)}{d\xi}\right|_{\xi=0} = \delta_{\beta 1}, \tag{54}$$

where

$$\int\int_\triangle dp\, dq \equiv \int_0^\infty dq \int_{|1-q|}^{1+q} dp,$$

β and γ represent {1,2,3}, and the summation convention is used for γ. The expressions of the functions U, V, and W are given in Appendix A.

The solution to Eqs. (53) and (54) can be written in the form

$$\overline{B}_\beta(\xi) = \overline{B}_\beta^0(\xi) + A_1\overline{B}_\beta^1(\xi) + A_2\overline{B}_\beta^2(\xi), \tag{55}$$

where $\overline{B}_\beta^0$ represents the homogeneous solutions of Eqs. (53) and (54) with $A_1 = A_2 = 0$, while $\overline{B}_\beta^\alpha$ represents the solutions of Eq. (53) for the initial conditions $\overline{B}_\beta(0) = 0$ and $d\overline{B}_\beta/d\xi(0) = 0$; $\overline{B}_\beta^1$ represents the solutions for $A_1 = 1$, $A_2 = 0$, and $\overline{B}_\beta^2$ represents the solutions for $A_1 = 0$, $A_2 = 1$. It can be shown that $U_{\beta 1} = U_{\beta 2}$, $U_{1\gamma} = U_{2\gamma}$, $V_1 = V_2$, and $W_1 = W_2$. Therefore, we have $\overline{B}_1^1 = \overline{B}_2^1$ and $\overline{B}_1^2 = \overline{B}_2^2$. However, $\overline{B}_1^0 \neq \overline{B}_2^0$ because of the differences between their initial conditions, see Eq. (54).

Figure 2 shows the function $B_\beta^\alpha(\xi)$ defined by $B_\beta^\alpha(\xi) \equiv \overline{B}_\beta^\alpha(\xi)G^{(0)}(\xi)$, where α and β represent {0,1,2} and {1,2,3}, respectively, and $\overline{B}_\beta^\alpha(\xi)$ is obtained numerically. (See Appendix B for details of the numerical methods.) The functions $B_\beta^\alpha(\xi)$ for all α and β decay with respect to the nondimensionalized time difference ξ for $\xi > 2$. Figure 2 suggests that $B_\beta^\alpha(\xi) \sim \exp(-c\xi)$ as $\xi \to \infty$, where c is a positive constant of order unity. This implies that $G_{ij}^S(\mathbf{k}, t+\tau, t) \sim \exp(-c\tau/T_L(k))$ for $\tau \to \infty$, because $G_{ij}^S(\mathbf{k}, t+\tau, t)$ is given by a linear combination of $T_L(k)B_\beta^\alpha(\tau/T_L(k))$. Thus, assumption (A-4) is consistent with the resulting G.

In general, $\mathcal{D}_{ij}[Q^S, G^S](\mathbf{k})$ may be written in the form

$$\mathcal{D}_{ij}[Q^S, G^S](\mathbf{k}) = D_{ijmn}(\mathbf{k})S_{mn}, \tag{56}$$

where $D_{ijmn}(\mathbf{k})$ is an isotropic fourth-order tensor and the time index t is suppressed for brevity. We therefore have from Eq. (39)

$$T_{ijmn}(\mathbf{k})S_{mn} = 0, \tag{57}$$

for any traceless S_{mn}, where

$$T_{ijmn}(\mathbf{k}) \equiv D_{ijmn}(\mathbf{k}) + K^{(0)}_{ijmn}(\mathbf{k}).$$

By setting $S_{mn} = \delta_{ma}\delta_{mb} - (1/3)\delta_{mn}\delta_{ab}$, one can show that

$$T_{ijab}(\mathbf{k}) - \tfrac{1}{3} T_{ijll}(\mathbf{k})\,\delta_{ab} = 0, \tag{58}$$

for any i, j, a, and b.

Equation (58) for various combinations of indices results in equations that are linear in A_1 and A_2, among which only two are linearly independent. This yields, for example, the following set of linearly independent equations for A_1 and A_2:

$$T_{ijij}(\mathbf{k}) - \tfrac{1}{3} T_{iijj}(\mathbf{k}) = 0, \tag{59}$$

$$\hat{k}_a \hat{k}_b T_{iiab}(\mathbf{k}) - \tfrac{1}{3} T_{iijj}(\mathbf{k}) = 0, \tag{60}$$

which may be written in the form

$$M\mathbf{A} + \mathbf{c} = \mathbf{0}, \tag{61}$$

where M is a constant 2×2 matrix, $\mathbf{A} = (A_1, A_2)^t$, $\mathbf{c}$ is a constant vector, and ^{t} denotes the transpose of the matrix or vector. By using numerical solutions for B^{α}_{β} and integrating numerically, we have

$$M = \begin{pmatrix} -3.11 & -0.738 \\ -3.39 & -0.225 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} -0.366 \\ -0.404 \end{pmatrix}. \tag{62}$$

The solution of Eq. (61) with Eq. (62) is given by

$$A_1 = -0.120 \pm 0.002, \quad A_2 = 0.009 \pm 0.014, \tag{63}$$

where the error estimates ± 0.002 and ± 0.014 are obtained by considering that there may be relative errors of roughly 1% in the numerical values in Eq. (62), as discussed in Appendix B. The Appendix also gives the details of the numerical methods used to integrate over wave vector space. Since the error estimate ± 0.014 in Eq. (63) for A_2 is large compared to its expected value 0.009, it is difficult to determine from the present calculations whether A_2 is identically 0 or small but finite. At present, no constraint that gives $A_2 = 0$ is known.

V. COMPARISON WITH DNS AND EXPERIMENTS

The tensor form of $Q^{(1)}$ in the DNS of IYK is consistent with Eqs. (43), (45), and (47) and the constants A_1 and A_2 are estimated to be

$$A_1 = -0.16 \pm 0.03, \quad A_2 = -0.40 \pm 0.06, \tag{64}$$

while the wind tunnel boundary layer experiments by Tsuji[8] give

$$A_1 \approx -0.17, \quad A_2 \approx -0.45, \tag{65}$$

where the shear rate S_{mn} in Eq. (43) is given by the local value of $S_{mn}(\mathbf{x}) = \partial \langle u_m(\mathbf{x}) \rangle / \partial x_n$, which may depend on the measurement position $\mathbf{x}$ (see the brief discussion in Sec. VII C).

In wind tunnel experiments and atmosphere observations, the one-dimensional cross spectrum $E^{(1\text{-D})}_{12}$ satisfying

$$\int_0^{\infty} dk_1 E^{(1\text{-D})}_{12}(k_1) = \langle u_1 u_2 \rangle,$$

has the similarity form

$$E^{(1\text{-D})}_{12}(k_1) = -C_1 \epsilon^{1/3} k_1^{-7/3} S, \tag{66}$$

where x_1 and x_2 are in the directions of the mean stream and velocity gradient, respectively, i.e., $S_{mn} \propto S\,\delta_{m1}\delta_{n2}$. According to the experiments by Wyngaard and Cote[6] (referred to as WC) and Saddoughi and Veeravalli[7] (referred to as SV),

$$C_1 \approx 0.14. \tag{67}$$

The $k_1^{-7/3}$ dependence is consistent with the present analysis, as well as with IYK's DNS and Tsuji's experiments. Equations (43), (45), and (47) imply that the constant C_1 is related to A_1 and A_2 as

$$C_1 = \frac{36\pi}{1729}(-33A_1 + 7A_2). \tag{68}$$

Substituting Eqs. (64) and (65) into Eq. (68) gives

$$C_1 = 0.16 \pm 0.07, \quad C_1 \approx 0.16, \tag{69}$$

respectively, which is in fairly good agreement with Eq. (67). The theoretical estimate from Eq. (63) gives

$$C_1 = 0.26 \pm 0.008. \tag{70}$$

The theoretical estimates Eqs. (63) and (70) are not in very good agreement with the DNS or experimental values, although the order of magnitude is similar.

Among the possible sources of the discrepancy between the theoretical estimate and DNS/experiments are

(0) the inadequacy of the LRA,
(i) the use of the simplifying assumptions (A-1)–(A-4),
(ii) finiteness of Re, and
(iii) the neglect of the homogeneous solution (zero modes) (Q^H, G^H). [See the discussion after Eqs. (50)–(52).]

Here (ii) implies that (ii-a) the scaling range, if it exists, can be only finite, and that (ii-b) the scaling itself may be influenced by the statistics outside the range if Re is not large enough. The validity of (i) may be affected by (ii), because $\delta(k)$ and $\mu(k)$ used in the assumptions (A-2) and (A-3) are k-dependent, so that the conditions in (A-2) and (A-3) may be not well satisfied in a wide enough range, if the scaling range is too narrow.

Regarding (ii), it is to be recalled that the theoretical LRA values given by Eqs. (63) and (70) are valid only for the asymptotic limit of Re$\to\infty$, i.e., the case in which the inertial subrange is infinitely or sufficiently wide, while the Reynolds number of the DNS in IYK is only modest ($R_\lambda = 284$). The experimental Reynolds number is only $R_\lambda \approx 420$ in Tsuji[8] and $R_\lambda \approx 1450$ in SV. (The Reynolds numbers of the experiments in WC are not given explicitly.) It is also to be noted that the slope of $Q^{(1)} \propto k^{-13/3}$ is much steeper than $Q^{(0)} \propto k^{-11/3}$ at small k, so that the integral in Eq. (A2) may be sensitive to the exact form of $Q^{(1)}$ at small k, or the width of the inertial subrange. A closer inspection of the integrals in Eq. (A2) shows that with substitution of the similarity forms Q^S and G^S of Eqs. (43)–(47) into $Q^{(1)}$ and $G^{(1)}$, the integral in Eq. (A2) does converge at small wavenumbers, but the convergence is much slower than that in Eq. (19) for isotropic Q [see the analysis after Eq. (72)]. Thus, it is not surprising that the inertial subrange in real turbulence with finite Re is sensitive to the exact form of the spectra at small wavenumbers, and in particular to the width of the inertial subrange.

Regarding (iii), the modification of the exponents of $Q^{(1)}$ and $E_{12}^{(1\text{-D})}$ from $-13/3$ and $-7/3$, respectively, may be significant, if the zero modes are not negligible. Similarly, it may be also significant, if (0) is not negligible. The above estimates of A_1, A_2, and C_1 from DNS and experiments are obtained by ignoring the modification. As discussed in Sec. I, it seems that the modification, if it exists, is small, and cannot be detected at present in a consistent way by experiments and DNS.

In order to obtain better estimates for $Q^{(1)}$ or A_1 and A_2 for real turbulence at finite Re, we need to improve the analysis. It would be interesting to take into account (0)–(iii) in the analysis, or to solve numerically the LRA equations as an initial value problem in the entire wavenumber range without assuming specific forms for $Q^{(1)}$ and $G^{(1)}$, by which one may avoid the problems associated with (i)–(iii). However, it is not easy to fully analyze the effects of any of (0)–(iii) or to solve numerically the LRA equations for anisotropic turbulence. In Sec. VI, we try to get some idea on the effect of (ii), especially (ii-a), by using a simple model.

VI. EFFECT OF THE FINITENESS OF THE WIDTH OF THE INERTIAL SUBRANGE

In order to get some idea on the effect of the finiteness of the width of the inertial subrange, let us consider a model spectrum in which $Q^{(1)}(\mathbf{k})$ and $G^{(1)}(\mathbf{k})$ outside the scaling range are simply discarded:

$$Q_{ij}^{(1)}(\mathbf{k},t)=\begin{cases} Q_{ij}^{S}(\mathbf{k},t), & (k_b \leqslant k \leqslant k_t), \\ 0, & (k<k_b, k>k_t), \end{cases} \tag{71}$$

$$G_{ij}^{(1)}(\mathbf{k},t+\tau,t)=\begin{cases} G_{ij}^{S}(\mathbf{k},t+\tau,t), & (k_b \leqslant k \leqslant k_t), \\ 0, & (k<k_b, k>k_t), \end{cases} \tag{72}$$

where k_b and k_t are the bottom and the top wavenumbers of the inertial subrange, respectively, Q^S is given by Eq. (43) with Eqs. (45) and (47), and G^S is given by Eq. (44) with Eqs. (46) and (47). This model will be referred to as the cutoff model. For simplicity, we assume the similarity forms (27) and (28) for $Q^{(0)}$ and $G^{(0)}$, respectively, throughout the entire wavenumber range. Then, $\mathcal{D}_{ij}[Q^{(1)},G^{(1)}](\mathbf{k})$ with Eqs. (71) and (72) can be written as

$$\mathcal{D}_{ij}[Q^{(1)},G^{(1)}](\mathbf{k})=D_{ijmn}(\mathbf{k};k_b,k_t)S_{mn}, \tag{73}$$

where $D_{ijmn}(\mathbf{k};k_b,k_t)$ is an isotropic fourth-order tensor.

Let $\zeta_b=k_b/k$ and $\zeta_t=k_t/k$. It can be shown that $D(\mathbf{k};\zeta_b k,k_t)-D(\mathbf{k};0,k_t)$ converges to 0 as $\sim\zeta_b^{2/3}$ in the limit of $\zeta_b\to 0$, while $D(\mathbf{k};k_b,\zeta_t k)-D(\mathbf{k};k_b,\infty)$ converges to 0 as $\sim\zeta_t^{-4/3}$ in the limit of $\zeta_t\to\infty$. Since the convergence of the former is slower, it is expected that a small wavenumber cutoff for $Q^{(1)}$ has a more significant effect on the dynamics of $Q^{(1)}$ than a large wavenumber cutoff. Thus, we will consider only a small wavenumber cutoff. Hereafter, we will set $k_t=\infty$, omit writing k_t explicitly, and denote ζ_b by ζ.

Equation (57) is now modified to

$$T_{ijmn}(\mathbf{k};k_b)S_{mn}=0, \tag{74}$$

where

$$T_{ijmn}(\mathbf{k};k_b)\equiv D_{ijmn}(\mathbf{k};k_b)+K_{ijmn}^{(0)}(\mathbf{k}).$$

As in the derivation of Eq. (61), we have linear equations of A_1 and A_2 that may be written in the form

$$M(\zeta)\mathbf{A}+\mathbf{c}=\mathbf{0}, \tag{75}$$

where $\mathbf{c}$ is the same as given in Eq. (62). The 2×2 matrix $M(\zeta)$ depends on $\mathbf{k}$ and k_b, but only through $\zeta\equiv k_b/k$. The ζ dependence of M implies that A_1 and A_2 may depend on k through ζ, i.e., $A_1=A_1(\zeta)$ and $A_2=A_2(\zeta)$, unlike the solutions for Eq. (61), and that the cutoff model of $(Q^{(1)},G^{(1)})$ does not satisfy Eq. (39) in a strict sense. [We assumed A_1, A_2 to be constant in Eq. (71).] This is an inevitable penalty of the cutoff model simplification. In the following analysis, we assume that the k dependence through ζ is weak, and that $a_1(k)$ and $a_2(k)$ may be approximated by Eq. (47) in which A_1 and A_2 are certain typical values of $A_1(\zeta)$ and $A_2(\zeta)$, respectively.

The matrix $M(\zeta)$ can be evaluated numerically for any $\zeta\geqslant 0$ by applying methods similar to those described in Appendix B. They may also be evaluated using their Taylor series expansions of the M matrix elements for small ζ. For example, if we discard terms of $o(\zeta^{2/3})$ in M, we have

$$M(\zeta)=\begin{pmatrix} -3.11-0.449\zeta^{2/3} & -0.738+1.91\zeta^{2/3} \\ -3.39+4.58\zeta^{2/3} & -0.225-1.37\zeta^{2/3} \end{pmatrix}. \tag{76}$$

In the following, we will denote the solutions of Eq. (75) with Eq. (76) as $(A_1^*(\zeta),A_2^*(\zeta))$ to distinguish them from the exact solutions of Eq. (75), $(A_1(\zeta),A_2(\zeta))$.

Figure 3 shows numerically estimated values of $A_1(\zeta)$ and $A_2(\zeta)$ for several ζ. It also shows the approximate values $A_1^*(\zeta)$ and $A_2^*(\zeta)$ for small ζ. It is evident that $A_1^*(\zeta)$ and

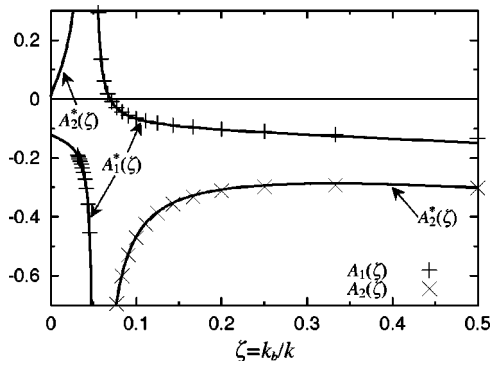

FIG. 3. Coefficients $A_1(\zeta)$ and $A_2(\zeta)$ of the anisotropic spectrum tensor in the cutoff model, as functions of $\zeta=k_b/k$. The symbols show the values computed from direct numerical integration. The lines show $A_1^*(\zeta)$ and $A_2^*(\zeta)$, the approximations of $A_1(\zeta)$ and $A_2(\zeta)$, respectively, for $\zeta\approx 0$.

$A_2^*(\zeta)$ agree well with $A_1(\zeta)$ and $A_2(\zeta)$ for $\zeta<0.5$. The implications of the figure may be summarized as follows:

(i) $A_1(\zeta)$ and $A_2(\zeta)$ agree well with the asymptotic values A_1 and A_2 for Re$\to\infty$ given by Eq. (63) only for $\zeta\ll 0.05$;
(ii) $A_1(\zeta)$ and $A_2(\zeta)$ are sensitive to ζ for $\zeta\sim 0.05$;
(iii) the dependence of $A_1(\zeta)$ and $A_2(\zeta)$ on ζ is weak for $0.1\lesssim\zeta\lesssim 0.5$; typical values in this range are $A_1(\zeta)\sim -0.1$ and $A_2(\zeta)\sim -0.3$;
(iv) $A_1(\zeta)$ decreases and $A_2(\zeta)$ increases with ζ over $0.1\lesssim\zeta\lesssim 0.4$.

Result (iii) implies that $a_1(k)/(\epsilon^{1/3}k^{-13/3})$ and $a_2(k)/(\epsilon^{1/3}k^{-13/3})$ may be approximated by the constants

$$A_1\sim -0.1, \quad A_1\sim -0.3, \tag{77}$$

respectively, when Re is moderate so that k_t/k_b is less than approximately 10. Furthermore, (iii) and (iv) suggest that A_1 and A_2 have weak Re dependence such that A_1 increases and A_2 decreases slowly with increasing Re over $3\lesssim k_t/k_b\lesssim 10$. Result (ii) indicates that a weak dependence of $A_1(\zeta), A_2(\zeta)$ on ζ is not well justified for $\zeta\sim 0.05$, and result (i) suggests that it would not be surprising if the asymptotic value for A_1 and A_2 could be achieved only by realizing a wide enough inertial subrange where k satisfying $\zeta=k_b/k<0.05$ can exist.

The value $z\equiv k_t/k_b$ in the DNS of IYK is estimated to be only 4.0, and it is less than 5.0 in Tsuji's experiments. Figure 1 (left) of WC and Fig. 19 (bottom) of SV suggest $z\sim 14$ and $z\sim 16$ for the corresponding experiments, respectively, provided that k_b and k_t are given by the bottom wavenumbers of the scaling ranges and the top wavenumbers of the spectra data given in the figures, respectively. (The bottom and the top wavenumbers of the similarity scaling range of $E_{12}^{(1\text{-D})}(k_1)$ are not given explicitly in WC and SV.) The above-mentioned analysis suggests that the inertial subranges of the DNS and experiments are too narrow, i.e., the Reynolds numbers are not large enough, to compare their $Q^{(1)}$ with the one calculated for the asymptotic limit of Re $\to\infty$, i.e., to compare the DNS or experimental values of A_1 and A_2 with those in Eq. (63) for Re$\to\infty$. The DNS and experimental values should be compared with the estimates for the limited width of the inertial subrange. In this respect, the estimates in Eq. (77) are in fairly good agreement with the DNS and experimental values in Eqs. (64) and (65).

To understand the anisotropy of small scale statistics at very large Re, it is desirable to realize or simulate turbulence with much larger Re. However, such realizations or simulations are not possible at present due to the limitations of the available facilities. In this context, it is worthwhile to note that we have recently performed a DNS of homogeneous turbulence for a simple mean shear flow using a spectral method that is free from alias error. The method of DNS is similar to that used by IYK, but the number of grid points $N^3=1024^3$ and Reynolds number $R_\lambda=480$, as well as the maximum wavenumber $K_{\max}=483$, are larger than those in IYK ($N^3=512^3$, $R_\lambda=284$ and $K_{\max}=241$).

The new DNS data are consistent with the Q^S spectrum derived in Sec. III as well as in IYK, and give

$$A_1=-0.15\pm 0.01, \quad A_2=-0.48\pm 0.02. \tag{78}$$

The value of A_1 in Eq. (78) is almost equal to that in Eq. (64), while the value of A_2 in Eq. (78) is slightly smaller than that in Eq. (64). The decrease of A_2 with Re is in agreement with result (iv) noted above. The weak dependence of A_1 on Re is also consistent with result (iii) and with the data shown in Fig. 3.

Recall that the estimate in Eq. (77) is based on the simplified cutoff model spectrum, Eqs. (71) and (72), which does not satisfy Eq. (39) in a strict sense as noted after Eq. (75). One might be interested in treating Q more realistically than the simple cutoff model, Eqs. (71) and (72), by using the DNS data of $Q^{(1)}$ for $k<k_b$ instead of discarding it. But, the analysis is lengthy and it turned out not to provide much improvement to the estimates of A_1 or A_2. It is therefore omitted from this paper. For further improvement of the estimate, one needs to develop better analytical or numerical treatment of the closure equations taking account of the discussion at the end of Sec. V. Such analytical or numerical treatment is left for a future problem.

VII. SUMMARY AND DISCUSSIONS

A. Summary

In this paper, we analyzed the anisotropic velocity correlation spectrum $Q^{(1)}$ in the inertial subrange of homogeneous turbulent shear flow by using the Lagrange renormalized approximation (LRA). The basic assumptions of the analysis are the fundamental symmetries (homogeneity and reflection invariance) of Q and G and the smallness of $\delta(k)=T_\tau(k)/T_S$ and $\mu(k)=T_\tau(k)/T_t(k)$ in the inertial subrange as described by (A-1)–(A-4). A theoretical estimate is given for the universal constants A_1 and A_2 that determine $Q^{(1)}$ and the universal functions $B_\beta^\alpha(\alpha=0,1,2;\beta=1,2,3)$ that determine $G^{(1)}$ in Sec. III.

The analysis in Sec. VI suggests that Re must be large enough so that $k_t/k_b\gg 20$ in order that $Q^{(1)}$ is approximated by its universal form in the limit of infinitely large Re. The analysis also gives rough estimates for $Q^{(1)}$ and its Re de-

pendence in turbulence at moderate Re. They are consistent with DNS and experiments.

B. Anisotropic components of the response function

To the authors' knowledge, the present study is the first attempt to analytically derive the anisotropic components of the Lagrangian response function $G^{(1)}$ using a spectral closure approximation without introducing any *ad hoc* adjusting parameters. One might hope that the correction $G^{(1)}$ is small and that $Q^{(1)}$ could be estimated by ignoring $G^{(1)}$. However, if $G^{(1)}$ is replaced by 0 in the Sec. IV analysis, we obtain

$$A_1 = -0.21 \pm 0.003, \quad A_2 = 0.58 \pm 0.02, \tag{79}$$

which are significantly different from the values given by Eq. (63). This shows the importance of treating $G^{(1)}$ properly instead of just discarding it.

C. Application to inhomogeneous turbulent shear flow

In this paper, we considered a simple mean shear flow, where the shear rate, i.e., $S_{mn}(\mathbf{x}) \equiv \partial \langle u_m(\mathbf{x}) \rangle / \partial x_n$, is constant in space and time for the sake of simplicity. However, this is not an essential assumption. The perturbational analysis of $Q(\mathbf{x},\mathbf{x}',t)$ and $G(\mathbf{x},t;\mathbf{x}',t')$ is applicable in principle even if S_{mn} depends on a position vector $\mathbf{x}$ and time t, provided that

$$\delta_\ell \equiv S_{\max} \epsilon^{-1/3} \ell^{2/3} \ll 1, \tag{80}$$

where $\ell = |\mathbf{x}-\mathbf{x}'|$ and $S_{\max}$ is the maximum S_{mn} in the flow domain and the time interval under consideration. Furthermore, we conjecture that if, in addition to Eq. (80), the characteristic length scale ℓ_S of S_{mn} is much larger than ℓ, then the leading order terms of $Q(\mathbf{x},\mathbf{x}',t)$ and $G(\mathbf{x},t;\mathbf{x}',t)$ in the perturbation expansion for small ℓ/ℓ_S are the homogeneous parts whose Fourier transforms are given by Eqs. (30) and (31), respectively. Here, $Q^{(1)}$ and $G^{(1)}$ are given by Q^S and G^S in Eqs. (43) and (44), respectively, and $S_{mn} = S_{mn}(\mathbf{x})$. Data from wind tunnel experiments[8] support this conjecture, since they are consistent with the tensor form of Eqs. (43), (45), and (47) when S_{mn} is replaced by the local value. Also, the measured constants A_1 and A_2 are in good agreement with those obtained by the DNS of homogeneous turbulent shear flow reported in IYK and in the present paper. The above-given conjecture could be examined in the framework of the LRA for inhomogeneous turbulence. This is left for future study.

D. Analyses based on the DIA

Previous studies of the anisotropic part $Q^{(1)}$ of the homogeneous turbulent shear flow spectrum, including those by Leslie,[33] Rubinstein *et al.*,[36] and Yoshizawa,[37] have used the Eulerian direct interaction approximation (DIA). In these studies, $Q^{(1)}$ is given in the following form:

$$Q_{ij}^{(1)}(\mathbf{k},t) = \int_{-\infty}^{t} ds G_{ia}^{E(0)}(\mathbf{k},t,s) C_{abmn}(\mathbf{k}) \times Q_{jb}^{E(0)}(-\mathbf{k},t,s) S_{mn} + (i \leftrightarrow j, \mathbf{k} \leftrightarrow -\mathbf{k}), \tag{81}$$

where $(i \leftrightarrow j, \mathbf{k} \leftrightarrow -\mathbf{k})$ denotes the term obtained by exchanging the indices i and j and also the wave vectors $\mathbf{k}$ and $-\mathbf{k}$ in the preceding term. [The comment on Leslie's $Q^{(1)}$ in IYK was incorrect; his $Q^{(1)}$ satisfies $Q_{ji}^{(1)}(\mathbf{k}) = Q_{ij}^{(1)}(\mathbf{k})$.] The functions $G^{E(0)}$ and $Q^{E(0)}$ are the Eulerian isotropic two-time response and velocity correlation spectra, respectively. [The definition of $G^{E(0)}$ and $Q^{E(0)}$ used by Yoshizawa is slightly different from those used in other studies, but, to the authors' understanding, his $G^{E(0)}$ and $Q^{E(0)}$ are also Eulerian response and correlation functions, because the velocity field used in the definitions obeys exactly the same equations as the Navier–Stokes equations (without the shear terms) in the Eulerian framework.] The symbol $C_{abmn}(\mathbf{k})$ denotes a nondimensional fourth-order tensor, the form of which differs in each study. It may contain differential operators with respect to $\mathbf{k}$ that act on the functions on the right-hand side. In the inertial subrange, $G_{ij}^{E(0)}(\mathbf{k})$ and $Q_{ij}^{E(0)}(\mathbf{k})$ are given by

$$G_{ij}^{E(0)}(\mathbf{k},t,s) = P_{ij}(\mathbf{k}) G^{E(0)}[\tau/T_{\mathrm{E}}(k)], \tag{82}$$

$$Q_{ij}^{E(0)}(\mathbf{k},t,s) = Q_{ij}^{(0)}(\mathbf{k},s) R[\tau/T_{\mathrm{E}}(k)], \tag{83}$$

where $G^{E(0)}(\cdot)$ and $R(\cdot)$ are nondimensional functions, $\tau \equiv t-s$, and $T_{\mathrm{E}}(k)$ is the Eulerian time scale of the eddies of size $\sim k^{-1}$.

It is assumed in the above-mentioned studies that $T_{\mathrm{E}}(k) \propto k^{-2/3}$. However, the original Eulerian DIA by Kraichnan[41] gives $T_{\mathrm{E}}(k) \propto k^{-1}$, and this scaling has been verified by several studies based on DNS analyses (see, for example, Ref. 42). Hence, it is difficult to justify the scaling $T_{\mathrm{E}}(k) \propto k^{-2/3}$. If one uses $Q_{ij}^{(0)}(\mathbf{k})$ given by Eq. (27) and $T_{\mathrm{E}}(k) \sim k^{-1}$ in the inertial subrange, then Eqs. (81)–(83) yield $Q^{(1)}(\mathbf{k}) \propto k^{-14/3}$, which contradicts the experiments by WC, SV, Tsuji, and the DNS reported in IYK and the present study.

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APPENDIX A: EXPRESSIONS OF FUNCTIONALS AND FUNCTIONS IN EQS. (34)–(36) AND (53)

The functionals $\mathcal{D}$, $\mathcal{J}$ and the function $\tilde{J}$ are given as follows:

$$\mathcal{D}_{ij}[Q^{(1)},G^{(1)}](\mathbf{k},t) = \mathcal{H}_{ij}[Q^{(1)},G^{(1)}](\mathbf{k},t) + \mathcal{H}_{ji}[Q^{(1)},G^{(1)}](-\mathbf{k},t), \tag{A1}$$

$$\mathcal{H}_{ij}[Q^{(1)},G^{(1)}](\mathbf{k},t)=\sum_{\mathbf{p},\mathbf{q}}^{\triangle}\int_0^{t-t_0}d\tau'(G^{(0)}_{af}(\mathbf{p},t+\tau',t)G^{(0)}_{bg}(\mathbf{q},t+\tau',t)G^{(0)}_{jh}(-\mathbf{k},t+\tau',t)P_{iab}(\mathbf{k})$$
$$\times\{Q^{(0)}_{gd}(\mathbf{q},t)[-P_{fde}(\mathbf{p})Q^{(1)}_{he}(-\mathbf{k},t)+\tfrac{1}{2}P_{hcd}(\mathbf{k})Q^{(1)}_{fc}(\mathbf{p},t)]$$
$$+Q^{(1)}_{gd}(\mathbf{q},t)[-P_{fde}(\mathbf{p})Q^{(0)}_{he}(-\mathbf{k},t)+\tfrac{1}{2}P_{hcd}(\mathbf{k})Q^{(0)}_{fc}(\mathbf{p},t)]\}$$
$$+\{G^{(0)}_{af}(\mathbf{p},t+\tau',t)G^{(0)}_{bg}(\mathbf{q},t+\tau',t)G^{(1)}_{jh}(-\mathbf{k},t+\tau',t)+G^{(0)}_{af}(\mathbf{p},t+\tau',t)G^{(1)}_{bg}(\mathbf{q},t+\tau',t)$$
$$\times G^{(0)}_{jh}(-\mathbf{k},t+\tau',t)+G^{(1)}_{af}(\mathbf{p},t+\tau',t)G^{(0)}_{bg}(\mathbf{q},t+\tau',t)G^{(0)}_{jh}(-\mathbf{k},t+\tau',t)\}$$
$$\times P_{iab}(\mathbf{k})Q^{(0)}_{gd}(\mathbf{q},t)[-P_{fde}(\mathbf{p})Q^{(0)}_{he}(-\mathbf{k},t)+\tfrac{1}{2}P_{hcd}(\mathbf{k})Q^{(0)}_{fc}(\mathbf{p},t)]), \tag{A2}$$

$$\mathcal{J}_{ij}[Q^{(1)},G^{(1)}](\mathbf{k},s+\tau,s)=-\sum_{\mathbf{p},\mathbf{q}}^{\triangle}\int_0^{\tau}d\tau' P_{ib}(-\mathbf{k})k_a\frac{p_bp_cp_l}{p^2}\times[G^{(0)}_{cd}(-\mathbf{q},s+\tau',s)Q^{(0)}_{da}(-\mathbf{q},s)G^{(1)}_{lj}(\mathbf{k},s+\tau,s)$$
$$+G^{(0)}_{cd}(-\mathbf{q},s+\tau',s)Q^{(1)}_{da}(-\mathbf{q},s)G^{(0)}_{lj}(\mathbf{k},s+\tau,s)$$
$$+G^{(1)}_{cd}(-\mathbf{q},s+\tau',s)Q^{(0)}_{da}(-\mathbf{q},s)G^{(0)}_{lj}(\mathbf{k},s+\tau,s)], \tag{A3}$$

$$\tilde{J}_{ij}(\mathbf{k},s+\tau,s)=-\sum_{\mathbf{p},\mathbf{q}}^{\triangle}\int_0^{\tau}d\tau'\ P_{ib}(-\mathbf{k})k_a\frac{p_bp_cp_l}{p^2}\Bigg\{G^{(0)}_{cd}(-\mathbf{q},s+\tau',s)Q^{(0)}_{da}(-\mathbf{q},s)\left[\frac{\partial}{\partial s}G^{(0)}_{lj}(\mathbf{k},s+\tau,s)\right]$$
$$+G^{(0)}_{cd}(-\mathbf{q},s+\tau,s)\left[\frac{\partial}{\partial s}Q^{(0)}_{da}(-\mathbf{q},s)\right]G^{(0)}_{lj}(\mathbf{k},s+\tau,s)$$
$$+\left[\frac{\partial}{\partial s}G^{(0)}_{cd}(-\mathbf{q},s+\tau',s)\right]Q^{(0)}_{da}(-\mathbf{q},s)G^{(0)}_{lj}(\mathbf{k},s+\tau',s)\Bigg\}. \tag{A4}$$

The functions $U_{\beta\gamma}(p,q,\xi)$, $V_\beta(\xi)$, and $W_\beta(\xi)$ where $\beta,\gamma=1,2,3$, are given as follows:

$$U_{1\gamma}(p,q,\xi)=\tfrac{1}{8}[3U^{\gamma}_{ijij}(\hat{\mathbf{k}},\mathbf{p},\mathbf{q};\xi)-U^{\gamma}_{ijji}(\hat{\mathbf{k}},\mathbf{p},\mathbf{q};\xi)$$
$$+\hat{k}_m\hat{k}_nU^{\gamma}_{iimn}(\hat{\mathbf{k}},\mathbf{p},\mathbf{q};\xi)-U^{\gamma}_{iijj}(\hat{\mathbf{k}},\mathbf{p},\mathbf{q};\xi)], \tag{A5}$$

$$U_{2\gamma}(p,q,\xi)=\tfrac{1}{8}[-U^{\gamma}_{ijij}(\hat{\mathbf{k}},\mathbf{p},\mathbf{q};\xi)+3U^{\gamma}_{ijji}(\hat{\mathbf{k}},\mathbf{p},\mathbf{q};\xi)$$
$$+\hat{k}_m\hat{k}_nU^{\gamma}_{iimn}(\hat{\mathbf{k}},\mathbf{p},\mathbf{q};\xi)-U^{\gamma}_{iijj}(\hat{\mathbf{k}},\mathbf{p},\mathbf{q};\xi)], \tag{A6}$$

$$U_{3\gamma}(p,q,\xi)=\tfrac{1}{8}[U^{\gamma}_{ijij}(\hat{\mathbf{k}},\mathbf{p},\mathbf{q};\xi)+U^{\gamma}_{ijji}(\hat{\mathbf{k}},\mathbf{p},\mathbf{q};\xi)$$
$$+7\hat{k}_m\hat{k}_nU^{\gamma}_{iimn}(\hat{\mathbf{k}},\mathbf{p},\mathbf{q};\xi)$$
$$-3U^{\gamma}_{iijj}(\hat{\mathbf{k}},\mathbf{p},\mathbf{q};\xi)], \tag{A7}$$

$$V_1(\xi)=\tfrac{1}{8}[3V_{ijij}(\hat{\mathbf{k}};\xi)-V_{ijji}(\hat{\mathbf{k}};\xi)$$
$$+\hat{k}_m\hat{k}_nV_{iimn}(\hat{\mathbf{k}};\xi)-V_{iijj}(\hat{\mathbf{k}};\xi)], \tag{A8}$$

$$V_2(\xi)=\tfrac{1}{8}[-V_{ijij}(\hat{\mathbf{k}};\xi)+3V_{ijji}(\hat{\mathbf{k}};\xi)$$
$$+\hat{k}_m\hat{k}_nV_{iimn}(\hat{\mathbf{k}};\xi)-V_{iijj}(\hat{\mathbf{k}};\xi)], \tag{A9}$$

$$V_3(\xi)=\tfrac{1}{8}[V_{ijij}(\hat{\mathbf{k}};\xi)+V_{ijji}(\hat{\mathbf{k}};\xi)$$
$$+7\hat{k}_m\hat{k}_nV_{iimn}(\hat{\mathbf{k}};\xi)-3V_{iijj}(\hat{\mathbf{k}};\xi)], \tag{A10}$$

$$W_1(\xi)=\tfrac{1}{8}[3W_{ijij}(\hat{\mathbf{k}};\xi)-W_{ijji}(\hat{\mathbf{k}};\xi)$$
$$+\hat{k}_m\hat{k}_nW_{iimn}(\hat{\mathbf{k}};\xi)-W_{iijj}(\hat{\mathbf{k}};\xi)], \tag{A11}$$

$$W_2(\xi)=\tfrac{1}{8}[-W_{ijij}(\hat{\mathbf{k}};\xi)+3W_{ijji}(\hat{\mathbf{k}};\xi)$$
$$+\hat{k}_m\hat{k}_nW_{iimn}(\hat{\mathbf{k}};\xi)-W_{iijj}(\hat{\mathbf{k}};\xi)], \tag{A12}$$

$$W_3(\xi)=\tfrac{1}{8}[W_{ijij}(\hat{\mathbf{k}};\xi)+W_{ijji}(\hat{\mathbf{k}};\xi)$$
$$+7\hat{k}_m\hat{k}_nW_{iimn}(\hat{\mathbf{k}};\xi)-3W_{iijj}(\hat{\mathbf{k}};\xi)], \tag{A13}$$

where

$$U^1_{ilmn}(\hat{\mathbf{k}},\mathbf{p},\mathbf{q};\xi)\equiv-\frac{K_o}{2}pqP_{ib}(-\hat{\mathbf{k}})\hat{k}_a\frac{p_bp_cp_l}{p^2}q^{-13/3}$$
$$\times G^{(0)}(q^{2/3}\xi)P_{da}(\mathbf{q})P_{cm}(\mathbf{q})P_{dn}(\mathbf{q}), \tag{A14}$$

$$U^2_{ilmn}(\hat{\mathbf{k}},\mathbf{p},\mathbf{q};\xi)\equiv-\frac{K_o}{2}pqP_{ib}(-\hat{\mathbf{k}})\hat{k}_a\frac{p_bp_cp_l}{p^2}q^{-13/3}$$
$$\times G^{(0)}(q^{2/3}\xi)P_{da}(\mathbf{q})P_{cn}(\mathbf{q})P_{dm}(\mathbf{q}), \tag{A15}$$

$$U^3_{ilmn}(\hat{\mathbf{k}},\mathbf{p},\mathbf{q};\xi)\equiv-\frac{K_o}{2}pqP_{ib}(-\hat{\mathbf{k}})\hat{k}_a\frac{p_bp_cp_l}{p^2}q^{-13/3}$$
$$\times G^{(0)}(q^{2/3}\xi)P_{da}(\mathbf{q})P_{cd}(\mathbf{q})\hat{q}_m\hat{q}_n, \tag{A16}$$

$$V_{ilmn}(\hat{\mathbf{k}};\xi) = -\sum_{\mathbf{p},\mathbf{q}}^{\triangle} P_{ib}(-\hat{\mathbf{k}})\hat{k}_a \frac{p_b p_c p_l}{p^2} \times q^{-13/3} G^{(0)}(q^{2/3}\xi) P_{cd}(\mathbf{q}) \times [P_{dm}(\mathbf{q})P_{an}(\mathbf{q}) + P_{dn}(\mathbf{q})P_{am}(\mathbf{q})], \tag{A17}$$

$$W_{ilmn}(\hat{\mathbf{k}};\xi) = -\sum_{\mathbf{p},\mathbf{q}}^{\triangle} P_{ib}(-\hat{\mathbf{k}})\hat{k}_a \frac{p_b p_c p_l}{p^2} q^{-13/3} \times G^{(0)}(q^{2/3}\xi) P_{cd}(\mathbf{q}) P_{da}(\mathbf{q}) \hat{q}_m \hat{q}_n. \tag{A18}$$

Here, $\hat{\mathbf{k}}$ is an arbitrary unit vector, and $(\mathbf{p},\mathbf{q})$ in Eqs. (A5)–(A18) are an arbitrary pair of vectors satisfying $|\mathbf{p}|=p$, $|\mathbf{q}|=q$, and $\mathbf{p}+\mathbf{q}=\hat{\mathbf{k}}$.

APPENDIX B: NUMERICAL METHODS

We obtained the functions $\overline{B}_\beta^\alpha$ in Eq. (55) numerically using a finite difference technique and an iteration method. In the following, α and β represent $\{0,1,2\}$ and $\{1,2,3\}$, respectively. We approximated the infinite interval $[0,\infty)$ of ξ by $N+1$ points $\xi_i = i\Delta$ $(i=0,1,\ldots,N)$ where $\Delta = \xi_{\max}/N$, and the functions $\overline{B}_\beta^\alpha(\xi)$ on the interval by the values of the functions at the N points, $\overline{B}_\beta^\alpha(\xi_i)$.

The zeroth approximations $\overline{B}_\beta^{\alpha(0)}$ of $\overline{B}_\beta^\alpha$ were given by their Taylor series expansions about $\xi=0$ to the second order:

$$\overline{B}_\beta^{0(0)}(\xi_i) = -\xi_i \delta_{\beta 1}, \quad \overline{B}_\beta^{1(0)}(\xi_i) = \tfrac{1}{2} V_\beta(0)\xi_i^2, \quad \overline{B}_\beta^{2(0)}(\xi_i) = \tfrac{1}{2} W_\beta(0)\xi_i^2. \tag{B1}$$

The $(m+1)$th approximations were obtained from the mth approximations in the following manner:

$$C_\beta^\alpha(\xi_i) = \overline{B}_\beta^{\alpha(m)}(\xi_i), \quad (i=0,1), \tag{B2}$$

$$C_\beta^\alpha(\xi_i) = 2C_\beta^\alpha(\xi_{i-1}) - C_\beta^\alpha(\xi_{i-2}) + \Delta^2 \Bigg[\int\int_{\triangle} dp\, dq\, U_{\beta\gamma}(p,q,\xi_{i-1}) \times \overline{B}_\gamma^{\alpha(m)}(q^{2/3}\xi_{i-1}) + \delta_{\alpha 1} A_1 V_\beta(\xi_{i-1}) + \delta_{\alpha 2} A_2 W_\beta(\xi_{i-1}) \Bigg], \quad (i=2,\ldots,N), \tag{B3}$$

$$\overline{B}_\beta^{\alpha(m+1)}(\xi_i) = \tfrac{1}{2}[\overline{B}_\beta^{\alpha(m)}(\xi_i) + C_\beta^\alpha(\xi_i)], \quad (i=1,\ldots,N), \tag{B4}$$

where $\overline{B}_\beta^{\alpha(m)}(\xi)$ at $\xi \neq \xi_i (i=0,1,\ldots,N)$ were defined by the linear interpolations of the points $(\xi_i, \overline{B}_\beta^{\alpha(m)}(\xi_i))$ for $\xi \leqslant \xi_{\max}$ and $\overline{B}_\beta^{\alpha(m)}(\xi) = \overline{B}_\beta^{\alpha(m)}(\xi_{\max})$ for $\xi > \xi_{\max}$.

We computed the p, q integrals on the right-hand side of Eq. (B3) by (i) symmetrizing the integrands with respect to p and q, (ii) integrating them numerically over the domain

$$\triangle' = \{(p,q) | q>0, \max(1-q,q) < p < 1+q\},$$

and then (iii) multiplying them by 2. We divided the domain $\triangle'$ into three subdomains, (a) $0<q<q_0$, (b) $q_0 \leqslant q \leqslant q_1$, and (c) $q_1 < q < \infty$, and performed the numerical integrations in each subdomain separately. Each numerical integration in domain (a) was performed as follows: (i) the integrations in the p direction for some fixed q were evaluated using a trapezoidal quadrature with variable transformations,[43] (ii) the data were fitted to a function of the form $c_1 q^{c_2}$, and then (iii) the fitted function was integrated analytically in the q direction. Domain (b) was further divided into additional subdomains; the integrations in each were evaluated using the trapezoidal quadrature with respect to both p and q. The integrals in domain (c) were simply neglected because the integrands decay as $G^{(0)}(q^{2/3}\xi) \propto \exp(-q^{2/3})$ for $q \to \infty$ and are therefore negligible if q_1 is sufficiently large.

We used

$$\xi_{\max}=4, \quad N=80 \ (\Delta=0.05), \quad m=10,$$

$$q_0 = 1/32, \quad q_1 = 1024,$$

in the present computation. The number of points in the p, q integrals were chosen so that the order of the relative numerical errors was at most $\sim 0.3\%$. The relative errors due to the finite differencing in the ξ direction were estimated to be $\sim 0.6\%$. The numerical errors due to the p, q integration and finite differencing in the ξ direction were estimated from the difference between the results of a computation with half of the resolution in either the p and q, or the ξ direction, respectively. The relative numerical errors due to the truncation of $B_\beta^\alpha(\xi)$ at $\xi_{\max}=4.0$ were estimated to be $\sim 0.1\%$ using the difference between the computations for $\xi_{\max}=4.0$ and $\xi_{\max}=8.0$, both with a lower resolution in ξ direction, $\Delta=0.2$. The relative error due to the iterations, $\sim 0.3\%$, was estimated from the difference between the results of the current iteration and those of the preceding iteration.

The p,q integrals in T of Eqs. (59) and (60) were computed in a manner similar to the one described above. The difference was that the integrations in domains (a) and (c) were performed with a fitting function, since the integrands decay $\sim q^{c_1}$ for $q \to 0$ and $\sim q^{-c_2}$ for $q \to \infty$, where c_1 and c_2 are positive constants. We chose $q_0 = 1/128$ and $q_1 = 32$ in the present computation. The numbers of points in the p,q integrals were chosen so that the order of the relative errors was at most 0.1%. The relative numerical errors due to the truncation of the universal functions $B_\beta^\alpha(\xi)$ $(\alpha=0,1,2,\ \beta=1,2,3)$ at $\xi=\xi_{\max}$ are $\sim 0.1\%$; these are estimated from the difference between the present results and those obtained using the extrapolated functions $B_\beta^{\alpha *}(\xi)$ defined by

$$B_\beta^{\alpha *}(\xi) = \begin{cases} B_\beta^\alpha(\xi), & (0 \leqslant \xi \leqslant \xi_{\max}), \\ \overline{B}_\beta^\alpha(\xi_{\max}) G^{(0)}(\xi), & (\xi_{\max} < \xi \leqslant 2\xi_{\max}), \\ 0, & (\xi > 2\xi_{\max}), \end{cases}$$

instead of $B_\beta^\alpha(\xi)$, which are truncated at $\xi=\xi_{\max}$.

Since all of the errors in the present numerical methods as well as the error introduced by the estimate of K_0 in Eq. (29) $(\lesssim 0.5\%)$ were smaller than 1%, we expect that the order of the relative errors in the numerical values given by Eq. (62) were smaller than $\sim 1\%$.

All the tensorial algebras and numerical computations were performed by using MATHEMATICA.

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