Inertial range structure of Gross-Pitaevskii turbulence within a spectral closure approximation

Kyo Yoshida and Toshihico Arimitsu

University of Tsukuba, 1-1-1 Tennoudai, Tsukuba, Ibaraki, 305-8571, Japan

E-mail: `yoshida.kyo.fu@u.tsukuba.ac.jp`

Abstract. The inertial range structure of turbulence obeying the Gross-Pitaevskii equation, the equation of motion for quantum fluids, is analyzed by means of a spectral closure approximation. It is revealed that, for the energy-transfer range, the spectrum of the order parameter field ψ obeys k^{-2} law for $k \ll k_*$ and k^{-1} law for $k \gg k_*$, where k_* is the wavenumber where the characteristic time scales associated with linear and nonlinear terms are of the same order. It is also shown that, for the particle-number-transfer range, the spectrum obeys k^{-1} law for $k \ll k_{*,\mathrm{n}}$ and $k^{-1/3}$ law for $k \gg k_{*,\mathrm{n}}$, where $k_{*,\mathrm{n}}$ is the wavenumber corresponds to k_* in the particle-number-transfer range.

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1. Introduction

The order parameter $\psi(x)$ ($x := \{\boldsymbol{x}, t\}$) for the condensed phase of a Bose gas satisfies the Gross-Pitaevskii (GP) equation [1, 2],

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(x) = -\frac{\hbar^2}{2m}\nabla^2\psi(x) - \mu\psi(x) + g|\psi(x)|^2\psi(x), \tag{1}$$

under a certain approximation (see, e.g., Ref. [3] for the derivation of the equation and the condition for the approximation). Here, $\hbar$ is the Planck constant divided by 2π, m the mass of the particles, μ the chemical potential, which is a constant throughout this paper, and g is the coupling constant. By introducing the velocity field $\boldsymbol{v}(x) := (\hbar/m)\nabla\varphi(x)$ with $\psi(x) = \sqrt{n(x)}\,\mathrm{e}^{i\varphi(x)}$, we can interpret the GP equation (1) as the equation of motion for a fluid as

$$\frac{\partial}{\partial t}n(x) = -\boldsymbol{\nabla}\cdot\boldsymbol{j}(x), \quad \boldsymbol{j}(x) := \frac{\hbar}{2mi}\left[\psi^*(x)\boldsymbol{\nabla}\psi(x) - \psi(x)\boldsymbol{\nabla}\psi^*(x)\right] = n(x)\boldsymbol{v}(x), \tag{2}$$

$$\frac{\partial}{\partial t}\boldsymbol{v}(x) = -\boldsymbol{v}(x)\cdot\boldsymbol{\nabla}\boldsymbol{v}(x) - \boldsymbol{\nabla}p(x), \quad p(x) := -\frac{\mu}{m} + \frac{gn(x)}{m} - \frac{\hbar^2}{2m^2}\frac{\nabla^2\sqrt{n(x)}}{\sqrt{n(x)}}, \tag{3}$$

where $n(x)$ is the number density of the condensate of quantum fluid. In comparison with the Navier-Stokes equation for the ordinary fluid, (3) has no dissipation term and the form of the last term in the r.h.s. is slightly different from the corresponding term $-\boldsymbol{\nabla}p(x)/\rho(x)$ in the Navier-Stokes equation where $\rho(x) := mn(x)$ is the mass density. The pressure-like term $p(x)$ depends not only on $n(x)$ but also on its spatial derivatives. The vorticity $\boldsymbol{\omega}(x) := \nabla\times\boldsymbol{v}(x)$ is $\mathbf{0}$ wherever $\boldsymbol{v}(x)$ is defined, i.e., $n(x) \neq 0$, and it is concentrate in lines where $n(x) = 0$. Due to the uniqueness of the phase $\varphi(x)$ up to modulo 2π, the circulation along a path around the line is quantized as $\oint_C \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{v}(x) = (2\pi\hbar/m)k$ where k is an integer.

The number of particles $\bar{n}$ and the energy $\bar{E}$ per unit volume are given by

$$\bar{n} := \frac{1}{V}\int \mathrm{d}\boldsymbol{x}|\psi(x)|^2, \tag{4}$$

$$\bar{E} := E_{\mathrm{K}}(t) + E_{\mathrm{I}}(t), \tag{5}$$

$$E_{\mathrm{K}}(t) := \frac{1}{V}\int \mathrm{d}\boldsymbol{x}\frac{\hbar^2}{2m}|\boldsymbol{\nabla}\psi(x)|^2 = \frac{1}{V}\int \mathrm{d}\boldsymbol{x}\,\left[\frac{1}{2}mn(x)|\boldsymbol{v}(x)|^2 + \frac{\hbar^2}{2m}|\boldsymbol{\nabla}\sqrt{n(x)}|^2\right], \tag{6}$$

$$E_{\mathrm{I}}(t) := \frac{1}{V}\int \mathrm{d}\boldsymbol{x}\,\frac{g}{2}|\psi(x)|^4 = \frac{1}{V}\int \mathrm{d}\boldsymbol{x}\frac{g}{2}[n(x)]^2, \tag{7}$$

where V is the volume of the whole integral domain, $E_{\mathrm{K}}(t)$ is the kinetic energy and $E_{\mathrm{I}}(t)$ is the interaction energy. Both $\bar{n}$ and $\bar{E}$ are the constants of the motion. The potential energy $\bar{E}_{\mathrm{P}} = -\mu\bar{n}$ is not included in the definition of the energy $\bar{E}$.

In experiments [4, 5] of turbulence of liquid ^{4}He in superfluid phase, there are some evidence suggesting that the spectrum corresponding to the velocity energy, the first term in the r.h.s. of (6), obeys a scaling law which is similar to the Kolmogorov energy spectrum $E(k) \propto k^{-5/3}$ in the ordinary fluid turbulence, where k is the wavenumber. Recall that superfluid phase is a mixture of superfluid and normal components and that the normal component remains even at very low temperature in experiments. Numerical simulation of GP equation provides a model for the ideal case of a pure superfluid component. Kobayashi and Tsubota [6] performed the numerical simulations with forcing and dissipation and obtained a spectrum which is consistent with the Kolmogorov spectrum for the spectrum corresponding to the velocity energy. This suggests the possibility that the superfluid component alone yields the Kolmogorov spectrum.

In the numerical simulations, the spectrum $F(k)$ related to $\psi(x)$ [see (67) for the definition] is also available since the data for the field $\psi(x)$ can be directly accessed. Note that $k^2F(k)$ is proportional to the the kinetic energy spectrum, i.e., the spectrum corresponding not only to the velocity energy but to the whole kinetic energy E_{K} in (6). Proment, Nazarenko and Onorato [7] (hereafter referred to as PNO) performed numerical simulations of GP equation with forcing and dissipation. The types and parameters of forcing and dissipation are varied among simulations and they observed $F(k) \propto k^{-1}$ when $\bar{n}$ is relatively small, cf. Fig. 3 of PNO, and $F(k) \propto k^{-2}$ when $\bar{n}$ is relatively large, cf. Fig. 4 of PNO. Yoshida and Arimitsu [8] observed a different scaling law $F(k) \propto k^{-2/3}$ in numerical simulations with a different type of forcing and dissipation from PNO. Thus, there is a possibility that the spectrum is sensitive to the manner of forcing and dissipation, especially when the scaling range is quite narrow as the present states of the numerical simulations. In order to avoid the effect of particular forms of the forcing and dissipation, we performed a preliminary numerical simulation of GP equation with the same setting as in Yoshida and Arimitsu but switching off the forcing and dissipation. The system without forcing and dissipation would finally relax to the thermal equilibrium state but the energy- or particle-number-transfer process, which is characteristics in turbulence, may be observed in the transient period. We observed $F(k) \propto k^{-2}$ which is consistent with the large $\bar{n}$ case of PNO in the transient period. We also observed a spectrum for the velocity energy that is consistent with $E(k) \propto k^{-5/3}$ in a higher wavenumber range. Since the scaling ranges are still narrow, this coexistence of the spectra $F(k) \propto k^{-2}$ and $E(k) \propto k^{-5/3}$ should be examined further. The details and further results of the new numerical simulation will be reported elsewhere.

When the nonlinear term, the last term in the r.h.s. of (1), is small compare to the other terms in the equation, the weak wave turbulence (WWT) theory [9, 10] can be used to derive the spectrum. The application of WWT theory to the GP equation was done by Dyachenko et al. [11] (hereafter referred to as DNPZ). For the energy-transfer range, the spectrum of ψ obeys the scaling law $F(k) \propto k^{-1}$ in the WWT theory. The scaling law coincides with that obtained in PNO for small $\bar{n}$. The effect of the nonlinear term in (1) may become dominant as $\bar{n}$ increases and thus the WWT theory is not capable of explaining the spectrum $F(k) \propto k^{-2}$ observed in PNO for large $\bar{n}$ or in our new simulation. We call the turbulence strong turbulence (ST), in contrast to WWT, when the nonlinear term is dominant.

The time evolution of the spectrum within the WWT region is discussed by Svistunov [12], where a self-similar form of $F(k,t)$ with the peak wavenumber $k_p(t)$ decreases with time is obtained. See also Semikoz and Tkachev [13] for the numerical and further study. The Laplacian in the first term of the r.h.s. of (1) implies multiplication of k^2 in Fourier space and thus the effect of the linear term is small in a low wavenumber region. The decrease of $k_p(t)$ suggests that the dominant part of the system would eventually enter the ST region.

In this paper, we attempt to derive theoretically the inertial range spectrum of the turbulence obeying GP equation not only for the WWT region but the ST region, by means of a spectral closure approximation, or in other words, a two-point closure approximation. As discussed by Kaneda [14], the closure equations are determined uniquely by the choice of quantities to close the equations under a fairly weak constraints. Our choice of the quantities are the two-point correlation function and response function of the field ψ. The reason for this choice is that it is the most simple and basic choice. This choice is essentially GP equation equivalent of the direct interaction approximation (DIA) [15] of the Navier-Stokes equation. A framework for the DIA of general wave equations with three-wave interaction can be found, e.g., in Ref. [16]. The GP equation have four-wave interaction and we will go further from the framework to obtain the structure of the similarity solution for the two-point correlation function, equivalently the spectrum.

This paper is organized as follows. In Sec. 2, the basic equations in Fourier space are given. In Sec. 3, the closure equations are given. In Secs. 4 and 5, the similarity solutions of the closure equations are analyzed for energy- and particle-number-transfer ranges, respectively. Discussion is given in Sec. 6.

2. Basic equations

Let us introduce a doublet by

$$\begin{pmatrix} \psi^{+}_{\boldsymbol{k}}(t) \\ \psi^{-}_{\boldsymbol{k}}(t) \end{pmatrix} := \mathrm{e}^{-L_{\boldsymbol{k}} t} \begin{pmatrix} \psi_{\boldsymbol{k}}(t) \\ \psi^{*}_{-\boldsymbol{k}}(t) \end{pmatrix}, \tag{8}$$

where $\psi_{\boldsymbol{k}}(t)$ is the Fourier transform of $\psi(\boldsymbol{x},t)$ with respect to $\boldsymbol{x}$,

$$\psi_{\boldsymbol{k}}(t) := \int \mathrm{d}\boldsymbol{x} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \psi(x), \tag{9}$$

and

$$L_{\boldsymbol{k}} := \mathrm{i}\left(-\frac{k^2}{2m} + \mu\right)\sigma_z, \tag{10}$$

with $\sigma_i (i = x, y, z)$ being the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

Let $n_{\boldsymbol{k}}(t)$ be the Fourier transform of $n(\boldsymbol{x},t)$ defined similarly as (9). $n_{\boldsymbol{k}}(t)$ can be written in terms of $\psi^{\alpha}_{\boldsymbol{k}}(t)$ as

$$n_{\boldsymbol{k}}(t) = \int_{\boldsymbol{pq}} \delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} O^{\beta\gamma}_{\boldsymbol{kpq}}(t)\psi^{\beta}_{\boldsymbol{p}}(t)\psi^{\gamma}_{\boldsymbol{q}}(t), \tag{12}$$

where $\int_{\boldsymbol{k}} = \sum_{\boldsymbol{k}}/V$ for a finite volume V (discrete wavevectors $\boldsymbol{k}$) and $\int_{\boldsymbol{k}} = \int \mathrm{d}^3\boldsymbol{k}/(2\pi)^3$ for $V \to \infty$ (continuous $\boldsymbol{k}$), $\int_{\boldsymbol{pq}}$ is the abbreviation for $\int_{\boldsymbol{p}}\int_{\boldsymbol{q}}$, $\delta_{\boldsymbol{k}} = (2\pi)^3\delta(\boldsymbol{k})$, and

$$O^{\beta\gamma}_{\boldsymbol{kpq}}(t) := \tilde{O}^{\beta'\gamma'}_{\boldsymbol{kpq}}(\mathrm{e}^{L_{\boldsymbol{p}}t})^{\beta'\beta}(\mathrm{e}^{L_{\boldsymbol{q}}t})^{\gamma'\gamma}, \qquad \tilde{O}^{\beta\gamma}_{\boldsymbol{kpq}} := \frac{1}{2}\sigma^{\beta\gamma}_{x}. \tag{13}$$

Here and hereafter, the upper Greece index denotes an element in $\{+,-\}$, and the repeated indices indicate the summation over the elements.

Equations (1) and (2) ($\hbar = 1$) can be rewritten in terms of $\psi^{\alpha}_{\boldsymbol{k}}(t)$ as

$$\frac{\partial}{\partial t}\psi^{\alpha}_{\boldsymbol{k}}(t) = g\int_{\boldsymbol{pqr}} \delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{r}} M^{\alpha\beta\gamma\zeta}_{\boldsymbol{kpqr}}(t)\psi^{\beta}_{\boldsymbol{p}}(t)\psi^{\gamma}_{\boldsymbol{q}}(t)\psi^{\zeta}_{\boldsymbol{r}}(t), \tag{14}$$

$$\frac{\partial}{\partial t}n_{\boldsymbol{k}}(t) = \int_{\boldsymbol{pq}} \delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} N^{\beta\gamma}_{\boldsymbol{kpq}}(t)\psi^{\beta}_{\boldsymbol{p}}(t)\psi^{\gamma}_{\boldsymbol{q}}(t), \tag{15}$$

where

$$M^{\alpha\beta\gamma\zeta}_{\boldsymbol{kpqr}}(t) := (\mathrm{e}^{-L_{\boldsymbol{k}}t})^{\alpha\alpha'}\tilde{M}^{\alpha'\beta'\gamma'\zeta'}_{\boldsymbol{kpqr}}(\mathrm{e}^{L_{\boldsymbol{p}}t})^{\beta'\beta}(\mathrm{e}^{L_{\boldsymbol{q}}t})^{\gamma'\gamma}(\mathrm{e}^{L_{\boldsymbol{r}}t})^{\zeta'\zeta}, \tag{16}$$

$$\tilde{M}^{\alpha\beta\gamma\zeta}_{\boldsymbol{kpqr}} := \begin{cases} -\frac{\mathrm{i}}{3} & \text{for } (\alpha,\beta,\gamma,\zeta) \in \{(+,-,+,+),(+,+,-,+),(+,+,+,-)\} \\ \frac{\mathrm{i}}{3} & \text{for } (\alpha,\beta,\gamma,\zeta) \in \{(-,+,-,-),(-,-,+,-),(-,-,-,+)\} \\ 0 & \text{otherwise} \end{cases}, \tag{17}$$

$$N^{\beta\gamma}_{\boldsymbol{kpq}}(t) := \tilde{N}^{\beta'\gamma'}_{\boldsymbol{kpq}}(\mathrm{e}^{L_{\boldsymbol{p}}t})^{\beta'\beta}(\mathrm{e}^{L_{\boldsymbol{q}}t})^{\gamma'\gamma}, \qquad \tilde{N}^{\beta\gamma}_{\boldsymbol{kpq}} := \frac{1}{4m}(p^2-q^2)\sigma^{\beta\gamma}_{y}. \tag{18}$$

The number density $\bar{n}$, the energy densities $E_{\mathrm{k}}(t)$ and $E_{\mathrm{I}}(t)$ are given in terms of $\psi^{\alpha}_{\boldsymbol{k}}(t)$ and $n_{\boldsymbol{k}}(t)$ as

$$\bar{n} = \frac{1}{V}\int_{\boldsymbol{k}}\psi^{+}_{\boldsymbol{k}}(t)\psi^{-}_{-\boldsymbol{k}}(t), \quad E_{\mathrm{K}}(t) = \frac{1}{V}\int_{\boldsymbol{k}}\frac{k^2}{2m}\psi^{+}_{\boldsymbol{k}}(t)\psi^{-}_{-\boldsymbol{k}}(t), \quad E_{\mathrm{I}}(t) = \frac{1}{V}\int_{\boldsymbol{k}}\frac{g}{2}n_{\boldsymbol{k}}(t)n_{-\boldsymbol{k}}(t). \tag{19}$$

Note that $\bar{n}$ does not depend on time since it is a constant of motion.

3. Closure equations

Let us assume statistical homogeneity in space, i.e., the statistical quantities are invariant under the spatial transition. Then, the two-point correlation function Q and the two-point response function G can be defined by

$$\langle\psi^{\alpha}_{\boldsymbol{k}}(t)\psi^{\beta}_{-\boldsymbol{k}'}(t')\rangle = Q^{\alpha\beta}_{\boldsymbol{k}}(t,t')\delta_{\boldsymbol{k}-\boldsymbol{k}'}, \tag{20}$$

$$\left\langle\frac{\delta\psi^{\alpha}_{\boldsymbol{k}}(t)}{\delta f^{\beta}_{\boldsymbol{k}'}(t')}\right\rangle = G^{\alpha\beta}_{\boldsymbol{k}}(t,t')\delta_{\boldsymbol{k}-\boldsymbol{k}'}, \tag{21}$$

where $\delta f_{\boldsymbol{k}}^{\alpha}(t)$ is an infinitesimal disturbance added to $\psi_{\boldsymbol{k}}^{\alpha}(t)$, and $\langle\cdot\rangle$ denotes an ensemble average.

The set of closed equations for $Q_{\boldsymbol{k}}(t,t')$ and $G_{\boldsymbol{k}}(t,t')$ can be obtained by the method of renormalized expansion and truncation. The method was applied to the Navier-Stokes equation by Kraichnan [17] and Kaneda [18]. The method is explained in the present context as follows.

(i) Expand Q and G in functional power series of the solutions $Q^{(0)}$ and $G^{(0)}$ for the zeroth-order in g.

(ii) Invert these expansions to obtain $Q^{(0)}$ and $G^{(0)}$ in functional power series of Q and G.

(iii) Substitute these inverted expansions into the primitive expansions of $\mathrm{d}Q/\mathrm{d}t$ and $\mathrm{d}G/\mathrm{d}t$ to obtain the renormalized expansions.

(iv) Truncate these renormalized expansions at the lowest nontrivial order.

Following the above procedure, we arrive at

$$\begin{aligned}
&\frac{\partial}{\partial t}Q_{\boldsymbol{k}}^{\alpha\beta}(t,t')\\
&= 3g\int_{\boldsymbol{p}} M_{\boldsymbol{kppk}}^{\alpha\gamma\zeta\eta}(t)Q_{\boldsymbol{p}}^{\gamma\zeta}(t,t)Q_{\boldsymbol{k}}^{\eta\beta}(t,t')\\
&\quad + g^2\int_{t_0}^{t}\mathrm{d}t''\int_{\boldsymbol{pqr}}\delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{r}}\Big[18M_{\boldsymbol{kpqr}}^{\alpha\gamma\zeta\eta}(t)M_{\boldsymbol{pqrk}}^{\lambda\nu\chi\kappa}(t'')G_{\boldsymbol{p}}^{\gamma\lambda}(t,t'')Q_{\boldsymbol{q}}^{\zeta\nu}(t,t'')Q_{\boldsymbol{r}}^{\eta\chi}(t,t'')Q_{\boldsymbol{k}}^{\kappa\beta}(t'',t')\\
&\qquad + 6M_{\boldsymbol{kpqr}}^{\alpha\gamma\zeta\eta}(t)M_{-\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{r}}^{\kappa\lambda\nu\chi}(t'')Q_{\boldsymbol{p}}^{\gamma\lambda}(t,t'')Q_{\boldsymbol{q}}^{\zeta\nu}(t,t'')Q_{\boldsymbol{r}}^{\eta\chi}(t,t'')G_{-\boldsymbol{k}}^{\beta\kappa}(t',t'')\Big]\\
&\quad + O(g^3),
\end{aligned}\tag{22}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}G_{\boldsymbol{k}}^{\alpha\beta}(t,t')\\
&= \delta^{\alpha\beta}\delta(t-t') + 3g\int_{\boldsymbol{p}} M_{\boldsymbol{kppk}}^{\alpha\gamma\zeta\eta}(t)Q_{\boldsymbol{p}}^{\gamma\zeta}(t,t)G_{\boldsymbol{k}}^{\eta\beta}(t,t')\\
&\quad + g^2\int_{t'}^{t}\mathrm{d}t''\int_{\boldsymbol{pqr}}\delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{r}}18M_{\boldsymbol{kpqr}}^{\alpha\gamma\zeta\eta}(t)M_{\boldsymbol{pqrk}}^{\lambda\nu\chi\kappa}(t'')G_{\boldsymbol{p}}^{\gamma\lambda}(t,t'')Q_{\boldsymbol{q}}^{\zeta\nu}(t,t'')Q_{\boldsymbol{r}}^{\eta\chi}(t,t'')G_{\boldsymbol{k}}^{\kappa\beta}(t'',t')\\
&\quad + O(g^3),
\end{aligned}\tag{23}$$

$$G_{\boldsymbol{k}}^{\alpha\beta}(t,t') = 0 \qquad (t<t'),\tag{24}$$

where t_0 is the initial time. We will take the limit $t_0\to-\infty$ in the following . The equations for Q and G are closed when the $O(g^3)$ terms in (22) and (23) are neglected. In the following, we will work within this closure approximation.

Let us assume that the statistical quantities are invariant under the global phase transformation, $\psi_{\boldsymbol{k}}^{\alpha}(t) \to \mathrm{e}^{\alpha\mathrm{i}\theta}\psi_{\boldsymbol{k}}^{\alpha}(t)$. Since $Q_{\boldsymbol{k}}^{\alpha\beta}(t,t')$ and $G_{\boldsymbol{k}}^{\alpha\beta}(t,t')$ transform as $Q_{\boldsymbol{k}}^{\alpha\beta}(t,t')\mathrm{e}^{(\alpha+\beta)\mathrm{i}\theta}$ and $G_{\boldsymbol{k}}^{\alpha\beta}(t,t')\mathrm{e}^{(\alpha-\beta)\mathrm{i}\theta}$ respectively under the global phase transformation, the correlation function and the response function can be written as

$$Q_{\boldsymbol{k}}^{+-}(t,t') = \mathrm{e}^{-2\mathrm{i}g\bar{n}(t-t')} Q_{\boldsymbol{k}}(t,t'), \qquad Q_{\boldsymbol{k}}^{-+}(t,t') = \mathrm{e}^{2\mathrm{i}g\bar{n}(t-t')} Q_{-\boldsymbol{k}}^{*}(t,t'), \tag{25}$$

$$G_{\boldsymbol{k}}^{++}(t,t') = \mathrm{e}^{-2\mathrm{i}g\bar{n}(t-t')} G_{\boldsymbol{k}}(t,t'), \qquad G_{\boldsymbol{k}}^{--}(t,t') = \mathrm{e}^{2\mathrm{i}g\bar{n}(t-t')} G_{-\boldsymbol{k}}^{*}(t,t'), \tag{26}$$

and otherwise 0. Here, the exponential factors are introduced to extract the time dependence of $Q_{\boldsymbol{k}}^{\alpha\beta}(t,t')$ and $G_{\boldsymbol{k}}^{\alpha\beta}(t,t')$ due to the terms of $O(g)$ in the r.h.s. of Eqs. (22) and (23) while the time dependence due to the terms of $O(g^2)$ are left to $Q_{\boldsymbol{k}}(t,t')$ and $G_{\boldsymbol{k}}(t,t')$. The equations for $Q_{\boldsymbol{k}}(t,t')$ and $G_{\boldsymbol{k}}(t,t')$ are given as

$$\begin{aligned}
&\frac{\partial}{\partial t} Q_{\boldsymbol{k}}(t,t') \\
=&g^2 \int_{-\infty}^{t} \mathrm{d}t'' \int_{\boldsymbol{pqr}} \delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{r}} \mathrm{e}^{\frac{\mathrm{i}}{2m}(k^2+p^2-q^2-r^2)(t-t'')} \\
&\times \Big[-2Q_{-\boldsymbol{p}}^{*}(t,t'') Q_{\boldsymbol{q}}(t,t'') G_{\boldsymbol{r}}(t,t'') Q_{\boldsymbol{k}}(t'',t') - 2Q_{-\boldsymbol{p}}^{*}(t,t'') G_{\boldsymbol{q}}(t,t'') Q_{\boldsymbol{r}}(t,t'') Q_{\boldsymbol{k}}(t'',t') \\
&\quad + 2G_{-\boldsymbol{p}}^{*}(t,t'') Q_{\boldsymbol{q}}(t,t'') Q_{\boldsymbol{r}}(t,t'') Q_{\boldsymbol{k}}(t'',t') + 2Q_{-\boldsymbol{p}}^{*}(t,t'') Q_{\boldsymbol{q}}(t,t'') Q_{\boldsymbol{r}}(t,t'') G_{\boldsymbol{k}}^{*}(t',t'') \Big],
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\frac{\partial}{\partial t} G_{\boldsymbol{k}}(t,t') \\
=&g^2 \int_{t'}^{t} \mathrm{d}t'' \int_{\boldsymbol{pqr}} \delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{r}} \mathrm{e}^{\frac{\mathrm{i}}{2m}(k^2+p^2-q^2-r^2)(t-t'')} \\
&\times \Big[-2Q_{-\boldsymbol{p}}^{*}(t,t'') Q_{\boldsymbol{q}}(t,t'') G_{\boldsymbol{r}}(t,t'') G_{\boldsymbol{k}}(t'',t') - 2Q_{-\boldsymbol{p}}^{*}(t,t'') G_{\boldsymbol{q}}(t,t'') Q_{\boldsymbol{r}}(t,t'') G_{\boldsymbol{k}}(t'',t') \\
&\quad + 2G_{-\boldsymbol{p}}^{*}(t,t'') Q_{\boldsymbol{q}}(t,t'') Q_{\boldsymbol{r}}(t,t'') G_{\boldsymbol{k}}(t'',t') \Big] + \delta(t-t'),
\end{aligned} \tag{28}$$

$$G_{\boldsymbol{k}}(t,t') = 0 \qquad (t < t'). \tag{29}$$

The equation for the one-time correlation function is given by

$$\frac{\partial}{\partial t} Q_{\boldsymbol{k}}(t,t) = \frac{\partial}{\partial t} Q_{\boldsymbol{k}}(t,t') \bigg|_{t'=t} + \text{c.c.}, \tag{30}$$

where c.c. denotes the complex conjugate.

The two-point correlation function for $n_{\boldsymbol{k}}(t)$ is introduced by

$$\langle n_{\boldsymbol{k}}(t) n_{-\boldsymbol{k}}(t') \rangle - \langle n_{\boldsymbol{k}}(t) \rangle \langle n_{-\boldsymbol{k}}(t') \rangle = Q_{\boldsymbol{k}}^{n}(t,t') \delta_{\boldsymbol{k}-\boldsymbol{k}'}, \tag{31}$$

and the equation for Q^n is given by

$$
\begin{aligned}
&\frac{\partial}{\partial t} Q^n_{\boldsymbol{k}}(t,t') \\
=&\mathrm{i}\int_{\boldsymbol{pq}} \delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} \frac{1}{2m}(p^2-q^2)\mathrm{e}^{\frac{\mathrm{i}}{2m}(p^2-q^2)(t-t')} Q^*_{-\boldsymbol{p}}(t,t')Q_{\boldsymbol{q}}(t,t') \\
&+g\int_{-\infty}^{t}\mathrm{d}t''\int_{\boldsymbol{pqrs}} \delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}}\delta_{\boldsymbol{k}-\boldsymbol{r}-\boldsymbol{s}}\frac{1}{m}(p^2-q^2)\mathrm{e}^{\frac{\mathrm{i}}{2m}\left[(-p^2+q^2)(t-t'')+(r^2-s^2)(t'-t'')\right]} \\
&\quad\times\Big[-G_{\boldsymbol{p}}(t,t'')Q^*_{-\boldsymbol{q}}(t,t'')Q^*_{\boldsymbol{r}}(t',t'')Q_{-\boldsymbol{s}}(t',t'') + Q_{\boldsymbol{p}}(t,t'')G^*_{-\boldsymbol{q}}(t,t'')Q^*_{\boldsymbol{r}}(t',t'')Q_{-\boldsymbol{s}}(t',t'') \\
&\qquad + Q_{\boldsymbol{p}}(t,t'')Q^*_{-\boldsymbol{q}}(t,t'')G^*_{\boldsymbol{r}}(t',t'')Q_{-\boldsymbol{s}}(t',t'') - Q_{\boldsymbol{p}}(t,t'')Q^*_{-\boldsymbol{q}}(t,t'')Q^*_{\boldsymbol{r}}(t',t'')G_{-\boldsymbol{s}}(t',t'')\Big] \\
&+O(g^2). \qquad (32)
\end{aligned}
$$

Hereafter, we will work within the approximation of neglecting the $O(g^2)$ terms. For the one-time correlation function, we have

$$
\frac{\partial}{\partial t} Q^n_{\boldsymbol{k}}(t,t) = \left.\frac{\partial}{\partial t} Q^n_{\boldsymbol{k}}(t,t')\right|_{t'=t} + \text{c.c.}. \qquad (33)
$$

The number density $\bar{n}$, the energy densities E_{K} and E_{I} are given in terms of Q and Q^n as

$$
\bar{n} = \int_{\boldsymbol{k}} Q_{\boldsymbol{k}}(t,t), \quad E_{\mathrm{K}}(t) = \int_{\boldsymbol{k}} \frac{k^2}{2m} Q_{\boldsymbol{k}}(t,t), \quad E_{\mathrm{I}}(t) = \frac{g}{2}\left(\int_{\boldsymbol{k}} Q^n_{\boldsymbol{k}}(t,t) + \bar{n}^2\right). \qquad (34)
$$

4. Energy-transfer range

In general, turbulence is a non-equilibrium state accompanied with an external forcing and dissipation. The forcing term $B_{\boldsymbol{k}}(t)$ and the dissipation term $D_{\boldsymbol{k}}(t)$ should be added to the r.h.s of (14). Let k_0 and k_1 be such wavenumbers that $B_{\boldsymbol{k}}(t)$ and $D_{\boldsymbol{k}}(t)$ can be neglected in comparison with the nonlinear term in the r.h.s. of (14) in the range $k_0 \ll k \ll k_1$, i.e., $k_0 \ll k \ll k_1$ is the inertial range. Since we will deal with the structure of turbulence within the inertial range, $B_{\boldsymbol{k}}(t)$ and $D_{\boldsymbol{k}}(t)$ will be neglected in the closure equations hereafter. We assume that the statistical quantities are isotropic, i.e., $Q_{\boldsymbol{k}}(t,t') = Q_k(t,t'), G_{\boldsymbol{k}}(t,t') = G_k(t,t')$ and $Q^n_{\boldsymbol{k}}(t,t') = Q^n_k(t,t')$, in the inertial range.

Let

$$
T_{\mathrm{L}}(k) := 2mk^{-2} \qquad (35)
$$

be the characteristic time scale associated with the linear term, the first term in the r.h.s. of (1). Note that μ does not play any role in the dynamics of Q and G since μ does not appear in (27) and (28). Let $T_{\mathrm{NL}}(k)$ be the time scale characterizing the change of $Q_k(t,t')$ and $G_k(t,t')$ with respect to $t-t'$. The explicit estimate of $T_{\mathrm{NL}}(k)$ will be given later.

Now, we consider the ST region, the wavenumber range within the inertial range where $T_{\mathrm{NL}}(k) \ll T_{\mathrm{L}}(k)$ is satisfied. In the ST region, we expect that the modes with wavenumber k lose their memory within the time scale $T_{\rm NL}(k)$, i.e., $Q_k(t,t'), G_k(t,t') \approx 0$ for $|t-t'| \gg T_{\rm NL}(k)$. Since $\mathrm{e}^{\frac{\mathrm{i}}{2m}k^2(t-t')}$ varies with the time scale $T_{\rm L}(k)$ ($\gg T_{\rm NL}(k)$), it may be approximated by its value at $t=t'$ in the time interval where $Q_k(t,t')$ and $G_k(t,t')$ take significant values. Thus, the factor $\mathrm{e}^{\frac{\mathrm{i}}{2m}(k^2+p^2-q^2-r^2)(t-t')}$ in the integrand of (27) and (28) can be approximated by 1. Assume that, for k in the range, the dominant contributions of the wavevector integrals in (27) and (28) come from the region where two of the norms p, q and r are of the same order as or smaller than k_0. Note that three norms p, q and r can not be smaller than k_0 simultaneously due to the factor $\delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{r}}$. The assumption will be verified from the result later. Under this assumption, we can use the approximation

$$\int_{\boldsymbol{k}} Q_k f(\boldsymbol{k}) \simeq f(\mathbf{0}) \int_{\boldsymbol{k}} Q_k, \tag{36}$$

for arbitrary function $f(\boldsymbol{k})$ which do not vary rapidly in the range $k \leq k_0$ and $f(\mathbf{0}) \neq 0$. Then, (27) and (28) reduce to

$$\frac{\partial}{\partial t} Q_k(t,t') = g^2 \int_{-\infty}^{t} \mathrm{d}t'' \Big[n(t,t'')\Big]^2 \Big[-4G_k(t,t'')Q_k(t',t'') + 6Q_k(t,t'')G_k(t',t'')\Big], \tag{37}$$

$$\frac{\partial}{\partial t} G_k(t,t') = -4g^2 \int_{t'}^{t} \mathrm{d}t'' \Big[n(t,t'')\Big]^2 G_k(t,t'')G_k(t'',t') + \delta(t-t'), \tag{38}$$

where

$$n(t,t') = \int_{\boldsymbol{k}} Q_k(t,t'), \tag{39}$$

and we have applied $Q_k^*(t,t') = Q_k(t,t')$ and $G_k^*(t,t') = G_k(t,t')$, which are now compatible with the equations.

Note that (38) with (29) implies that $G_k(t,t')$ satisfies the same equation and the boundary condition for all k. Thus, $G_k(t,t')$ do not depend on k. Let us write $Q_k(t,t')$ as

$$Q_k(t,t') = R_k(t,t')Q_k(t',t'), \tag{40}$$

with $R_k(t,t)=1$. In the following, we consider the statistically stationary states, in which $Q_k(t,t')$ and $G_k(t,t')$ depend on t and t' only through $t-t'$. Especially, we denote $Q_k(t,t)$ by Q_k. From (37) and (40), we see that $R_k(t,t')$ also do not depend on k. For rescaled functions $\hat{G}$ and $\hat{R}$, specified by

$$G_k(t,t') = \hat{G}(g\bar{n}\ (t-t')), \quad R_k(t,t') = \hat{R}(g\bar{n}\ (t-t')), \tag{41}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{R}(\tau) = \int_{-\infty}^{\tau} \mathrm{d}\tau' \Big[\hat{R}(\tau-\tau')\Big]^2 \Big[-2\hat{G}(\tau-\tau')\hat{R}(\tau') + 6\hat{R}(\tau-\tau')\hat{G}(-\tau')\Big], \tag{42}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{G}(\tau) = -4\int_{0}^{\tau} \mathrm{d}\tau' \Big[\hat{R}(\tau-\tau')\Big]^2 \hat{G}(\tau-\tau')\hat{G}(\tau') + \delta(\tau), \tag{43}$$

$$\hat{G}(\tau) = 0 \qquad (\tau < 0). \tag{44}$$

The scaling in (41) implies that

$$T_{\mathrm{NL}}(k) = g^{-1}\bar{n}^{-1}, \tag{45}$$

that is, $T_{\mathrm{NL}}(k)$ is independent of k. Since $T_{\mathrm{NL}}(k)$ is related to the time scale of the loss of memory for the modes with wavenumber k and not the inverse of the angular frequency $\omega(k)$, $T_{\mathrm{NL}}(k)$ being independent of k does not imply the strong correlation among different wavenumber modes due to a phase lock. The wavenumber k_* at which $T_{\mathrm{NL}}(k_*) = T_{\mathrm{L}}(k_*)$ is given by

$$k_* = (2m)^{1/2} g^{1/2} \bar{n}^{1/2}. \tag{46}$$

Provided that $k_0 \ll k_*$, (41) with (42)–(44) is valid in $k_0 \ll k \ll \min(k_*, k_1)$.

The energy flux $\Pi(K)$ into modes with wavenumber larger than K is defined by

$$\Pi(K) := \frac{\partial}{\partial t} \int_{\substack{\boldsymbol{k}\\ k>K}} \left[\frac{k^2}{2m} Q_{\boldsymbol{k}}(t,t) + \frac{g}{2} Q^n_{\boldsymbol{k}}(t,t) \right]. \tag{47}$$

From (27),(30), (32), (33) and (47),we have

$$\begin{aligned}
\Pi(K) = g^2 &\left(\int_{\substack{\boldsymbol{kpqr}\\ k>K}} - \int_{\substack{\boldsymbol{kpqr}\\ |\boldsymbol{k}-\boldsymbol{q}|>K, |\boldsymbol{p}-\boldsymbol{r}|>K}} \right) \delta_{\boldsymbol{k}+\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{r}} \int_{-\infty}^{t} \mathrm{d}t' \mathrm{e}^{\frac{\mathrm{i}}{2m}(k^2+p^2-q^2-r^2)(t-t')} \\
&\times \frac{k^2}{m} \Big[-Q^*_k(t,t') Q^*_p(t,t') Q_q(t,t') G_r(t,t') - Q^*_k(t,t') Q^*_p(t,t') G_q(t,t') Q_r(t,t') \\
&\qquad + Q^*_k(t,t') G^*_p(t,t') Q_q(t,t') Q_r(t,t') + G^*_k(t,t') Q^*_p(t,t') Q_q(t,t') Q_r(t,t') \Big] \\
&+ \text{c.c.}.
\end{aligned} \tag{48}$$

Assume that, for K in the inertial range, the dominant contribution in the wavevector integrals in (48) comes from the region where one of the norms k, p, q and r is of the same order as or smaller than k_0. The assumption will be verified from the result later. By applying $\mathrm{e}^{\frac{\mathrm{i}}{2m}(k^2+p^2-q^2-r^2)(t-t')} = 1$ and (36) to (48), we obtain

$$\begin{aligned}
\Pi(K) = 4g^2 &\left(\int_{\substack{\boldsymbol{kpq}\\ k>K, p<K}} - \int_{\substack{\boldsymbol{kpq}\\ k<K, p>K}} \right) \delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} \frac{k^2}{m} \\
&\times \int_{-\infty}^{t} \mathrm{d}t' n(t,t') Q_p(t,t') \Big[Q_q(t,t') G_k(t,t') - G_q(t,t') Q_k(t,t') \Big],
\end{aligned} \tag{49}$$

where we have applied $Q^*_k(t,t') = Q_k(t,t')$ and $G^*_k(t,t') = G_k(t,t')$ since they are compatible with (37) and (38). Note that (36) can not be applied to (49) since $f(\mathbf{0}) = 0$.

In the energy-transfer range, the energy flux $\Pi(K)$ is a constant Π which is independent of K. Let as assume the similarity form

$$Q_k = A k^a (2m)^b g^c \bar{n}^d |\Pi|^e, \tag{50}$$

where A is a constant, in the range $k_0 \ll k \ll \min(k_*, k_1)$. By using

$$\int_{\boldsymbol{p},\boldsymbol{q}} \delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} f(k,p,q) = \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}p \int_{|k-p|}^{k+p} \mathrm{d}q \frac{pq}{k} f(k,p,q), \tag{51}$$

for an arbitrary function $f(k,p,q)$ which depends only on the norms k,p and q, and the change of the variables $\{k,p,q\} \to \{\alpha, p', q'\}$ given by $\alpha = K/k$, $p' = p/k$ and $q' = q/k$ to (49), we have

$$a = -4, \quad b = \frac{1}{2}, \quad c = -\frac{1}{2}, \quad d = 0, \quad e = \frac{1}{2}, \quad A = |I_1|^{-1/2}, \tag{52}$$

with

$$\begin{aligned} I_1 &:= \frac{64\pi^2}{(2\pi)^6}\left(\int_0^1 \frac{\mathrm{d}\alpha}{\alpha}\int_0^\alpha \mathrm{d}p' - \int_1^\infty \frac{\mathrm{d}\alpha}{\alpha}\int_\alpha^\infty \mathrm{d}p'\right)\int_{|1-p'|}^{1+p'} \mathrm{d}q' p' q' \left[p'^{-4}q'^{-4} - p'^{-4}\right] \\ &\quad \times \int_0^\infty \mathrm{d}\tau \left[\hat{R}(\tau)\right]^3 \hat{G}(\tau) \\ &= \frac{3}{4\pi^2}\int_0^\infty \mathrm{d}\tau \left[\hat{R}(\tau)\right]^3 \hat{G}(\tau). \end{aligned} \tag{53}$$

(50) and (52) imply

$$Q_k = |I_1|^{-1/2}(2m)^{1/2} g^{-1/2} |\Pi|^{1/2} k^{-4}, \tag{54}$$

for $k_0 \ll k \ll \min(k_*, k_1)$. The assumptions on the dominant contribution of the integrals in (27), (28) and (48) are valid when $a < -3$. Thus, the assumptions are consistent with the obtained result (52). We have $\mathrm{sgn}(\Pi) = \mathrm{sgn}(I_1)$. Since $\hat{R}(0) = \hat{G}(0) = 1$ and we expect $\hat{R}(\tau), \hat{G}(\tau) \to 0$ for $\tau \to \infty$, it is likely that the main contribution in the τ integral in (53) comes from small τ and thus $I_1 > 0$ and $\Pi > 0$, i.e., the energy is transferred from low to high wavenumbers.

Note that the constant energy flux implies

$$\frac{\partial}{\partial t}\left[\frac{k^2}{2m} Q_{\boldsymbol{k}}(t,t) + \frac{g}{2} Q^n_{\boldsymbol{k}}(t,t)\right] = 0, \tag{55}$$

by definition (47). We see from (30),(32),(33) and (37) that

$$\frac{k^2}{2m}\frac{\partial}{\partial t} Q_k(t,t) = -\frac{g}{2}\frac{\partial}{\partial t} Q^n_k(t,t) = \frac{2}{m} g^2 k^2 \int_{-\infty}^t \mathrm{d}t'' \left[n(t,t'')\right]^2 Q_k(t,t''), \tag{56}$$

which is not 0 in general and likely to be positive from the same reason noted for I_1. This implies that the energy is flowing from the interaction energy to the kinetic energy and that the assumption of statistical stationarity is violated. Therefore, some correction should be made on Q_k in (54) to maintain the statistical stationarity.

The situation is somewhat similar to that of the magnetohydrodynamic (MHD) turbulence. In the closure analysis of MHD turbulence [19], the equipartition between the kinetic and magnetic energy spectra yields energy flow from the kinetic to magnetic energy. The nonzero residual energy spectrum, difference between the kinetic and magnetic energy spectrum, is introduce to cancel the energy flow and to maintain the statistical stationarity. However, the present situation and that in the MHD turbulence are not completely parallel since velocity and magnetic fields are mutually independent degrees of freedom in MHD but $\psi(x)$ and $n(x) = |\psi(x)|^2$ are not so. Therefore, it is not yet clear whether the statistical stationary can be maintained by introducing a some sort of residual energy spectrum. It is left for a future study to derive the correction of Q_k to maintain the statistical stationarity.

We now consider the WWT region, the wavenumber range within the inertial range where non-linearity is weak in the sense $T_{\rm NL}(k) \gg T_{\rm L}(k)$. The wavenumber range is located at $\max(k_*, k_0) \ll k \ll k_1$. In this range, $Q_k(t,t')$ and $G_k(t,t')$ in the integrand of (48) can be approximated by Q_k and 1, respectively. By virtue of this approximation and the identity

$$\int_0^\infty \mathrm{d}t e^{\mathrm{i}\omega t} = -\pi\delta(\omega) + \mathrm{p.v.}\frac{1}{\mathrm{i}\omega}, \tag{57}$$

where p.v. denotes the principal value, (48) reads

$$\begin{aligned}\Pi(K) = 2g^2 \int_{\substack{\boldsymbol{kpqr}\\ k>K}} \delta_{\boldsymbol{k}+\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{r}} \pi\delta\left(\frac{1}{2m}(k^2+p^2-q^2-r^2)\right) \\ \times \frac{k^2}{m} Q_k Q_p Q_q Q_r (-Q_k^{-1} - Q_p^{-1} + Q_q^{-1} + Q_r^{-1}).\end{aligned} \tag{58}$$

Note that the contribution of the interaction energy to $\Pi(K)$, i.e., the second term in the r.h.s. of (47) or the integral over the domain where $|\boldsymbol{k}-\boldsymbol{q}| > K$ and $|\boldsymbol{p}-\boldsymbol{r}| > K$ in the r.h.s. of (48), vanishes in (58) as shown in the following. Apply three types of change of variables $\{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}\} \to \{\boldsymbol{p},\boldsymbol{k},\boldsymbol{r},\boldsymbol{q}\}, \{\boldsymbol{q},\boldsymbol{r},\boldsymbol{k},\boldsymbol{p}\}$ and $\{\boldsymbol{r},\boldsymbol{q},\boldsymbol{p},\boldsymbol{k}\}$ which do not change the domain of the integral, $|\boldsymbol{k}-\boldsymbol{q}| > K$ and $|\boldsymbol{p}-\boldsymbol{r}| > K$, to the corresponding integral in (48) to obtain three alternative expressions for the integral. Taking the average over the original and three alternative expressions of the integral yields the factor $(k^2+p^2-q^2-r^2)\delta\left(\frac{1}{2m}(k^2+p^2-q^2-r^2)\right) = 0$. Since contribution from the interaction energy vanishes, (58) is equivalent to the corresponding equation of the WWT theory. The similarity solution of Q_k for the present case was already obtained in DNPZ. Here, we rederive it in the present context.

By substituting (50) into (58), we obtain

$$a = -3, \quad b = 0, \quad c = -\frac{2}{3}, \quad d = 0, \quad e = \frac{1}{3}. \tag{59}$$

However, for $a = -3$, the wavevector integrals in (58) are logarithmically divergent as $k_{\rm b} \to 0$, where $k_{\rm b} := \max(k_*, k_0)$ is the bottom wavenumber of the scaling range. Therefore, we have to work with the cutoff of the spectrum at $k_{\rm b}$ explicitly. The wavevector integrals are convergent for $k_1 \to \infty$. These imply that, for $K \gg k_{\rm b}$, the integral over the wavenumber larger than K is negligible in comparison with that over the wavenumber smaller than K with respect to each of $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$. By using (36) to the integrals of (58) over $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$ satisfying p or q or $r < K$, we obtain

$$\begin{aligned}\Pi(K) = 2g^2 \int_{\substack{\boldsymbol{r}\\ r<K}} Q_r \int_{\substack{\boldsymbol{kpq}\\ k>K}} \frac{k^2}{m} Q_k Q_p Q_q \\ \times \Big[\delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} \pi\delta\left(\frac{1}{2m}(k^2 - p^2 - q^2)\right)(-Q_k^{-1} + Q_p^{-1} + Q_q^{-1}) \\ + \delta_{\boldsymbol{k}-\boldsymbol{p}+\boldsymbol{q}} \pi\delta\left(\frac{1}{2m}(k^2 - p^2 + q^2)\right)(-Q_k^{-1} + Q_p^{-1} - Q_q^{-1}) \\ + \delta_{\boldsymbol{k}+\boldsymbol{p}-\boldsymbol{q}} \pi\delta\left(\frac{1}{2m}(k^2 + p^2 - q^2)\right)(-Q_k^{-1} - Q_p^{-1} + Q_q^{-1})\Big]. \end{aligned} \tag{60}$$

Let us assume the modified form of the spectrum

$$Q_k = A g^{-2/3} \Pi^{1/3} k^{-3} \phi(k), \tag{61}$$

for $k \geq k_{\mathrm{b}}$ and 0 otherwise, where $\phi(k)$ is a slowly varying function which is to be determined. When (61) is substituted into (60), ϕ in the integrals can be approximated by the typical value $\phi(K)$. Then, we have

$$\phi(k) = \left(\ln \frac{k}{k_{\mathrm{b}}}\right)^{-1/3}, \qquad A = \left(\frac{I_2}{2\pi^2}\right)^{-1/3}, \tag{62}$$

where

$$\begin{aligned} I_2(a) := 4\int_{\substack{\boldsymbol{kpq}\\ k>K}} k^2 (kpq)^a \Big[\delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} \pi\delta(k^2 - p^2 - q^2)(-k^{-a} + p^{-a} + q^{-a}) \\ + \delta_{\boldsymbol{k}-\boldsymbol{p}+\boldsymbol{q}} \pi\delta(k^2 - p^2 + q^2)(-k^{-a} + p^{-a} - q^{-a}) \\ + \delta_{\boldsymbol{k}+\boldsymbol{p}-\boldsymbol{q}} \pi\delta(k^2 + p^2 - q^2)(-k^{-a} - p^{-a} + q^{-a})\Big], \end{aligned} \tag{63}$$

and $I_2 := I_2(-3)$. It can be proved that I_2 is independent of K and positive, which implies $\Pi > 0$, i.e., the energy is transferred from low to high wavenumber. The proof of $I_2 > 0$ is given in Appendix A.

Within WWT region, let us consider another situation such that sufficient amplitude of mode is present in wavenumbers smaller than k_{b} and that

$$\int_{\substack{\boldsymbol{r}\\ r<K}} Q_r \simeq \bar{n} \tag{64}$$

is satisfied for $K > k_{\mathrm{b}}$. By substituting (50) and (64) into (60), we obtain

$$a = -3, \quad b = 0, \quad c = -1, \quad d = -\frac{1}{2}, \quad e = \frac{1}{2}, \quad A = I_2^{-1/2}, \tag{65}$$

which imply

$$Q_k = I_2^{-1/2} g^{-1} \bar{n}^{-1/2} \Pi^{1/2} k^{-3}. \tag{66}$$

As shown in Appendix A, we have $I_2 > 0$ and therefore $\Pi > 0$, i.e., energy is transferred from low to high wavenumber. Unlike (61), there is no logarithmic correction in (66). The relation between (61) and (66) is similar to that between $k^{-3}[\ln(k/k_\mathrm{b})]^{-1/3}$ energy spectrum by Kraichnan [20] and k^{-3} energy spectrum by Kaneda and Ishihara [21] in the enstrophy-transfer range of the two-dimensional Navier-Stokes turbulence.

The results for the energy-transfer range are summarized in terms of one-dimensional spectrum

$$F(k) = \int_{\boldsymbol{k}'} \delta(k' - k) Q_{\boldsymbol{k}'}, \tag{67}$$

as follows. In the ST region, $k_0 \ll k \ll \min(k_*, k_1)$,

$$F(k) = C_1 (2m)^{1/2} g^{-1/2} |\Pi|^{1/2} k^{-2}, \tag{68}$$

with $C_1 = |I_1|^{-1/2}/(2\pi^2)$ and some sort of correction is necessary to maintain the statistical stationarity. The correction is not known yet. The direction of the energy-transfer is likely to be forward, i.e., $\Pi > 0$. In WWT region, $\max(k_*, k_0) \ll k \ll k_1$,

$$F(k) = C_2 g^{-2/3} \Pi^{1/3} k^{-1} \left(\ln \frac{k}{k_\mathrm{b}} \right)^{-1/3}, \tag{69}$$

with $C_2 = (4\pi^4 I_2)^{-1/3}$ for the case (64) is not satisfied, and

$$F(k) = C_2' g^{-1} \bar{n}^{-1/2} \Pi^{1/2} k^{-1} \tag{70}$$

with $C_2' = I_2^{-1/2}/(2\pi^2)$ for the case (64) is satisfied. The direction of the energy-transfer is forward, i.e., $\Pi > 0$, for both (69) and (70). Note that the spectra (68) and (70) can coexist with (68) located at the lower wavenumber range and (70) the higher.

5. Particle-number-transfer range

The flux $\Pi_\mathrm{n}(K)$ of the particle number into modes with wavenumber larger than K is given by

$$\Pi_\mathrm{n}(K) := \frac{\partial}{\partial t} \int_{\substack{\boldsymbol{k} \\ k>K}} Q_{\boldsymbol{k}}(t,t). \tag{71}$$

From (27),(30) and (71), we obtain

$$\begin{aligned} \Pi_\mathrm{n}(K) = 2g^2 \int_{\substack{\boldsymbol{kpqr} \\ k>K, q<K}} \delta_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{r}} \int_{-\infty}^{t} \mathrm{d}t' \mathrm{e}^{\frac{\mathrm{i}}{2m}(k^2+p^2-q^2-r^2)(t-t')} \\ \times \Big[-Q_p^*(t,t')Q_q(t,t')G_r(t,t')Q_k^*(t,t') - Q_p^*(t,t')G_q(t,t')Q_r(t,t')Q_k^*(t,t') \\ + G_p^*(t,t')Q_q(t,t')Q_r(t,t')Q_k^*(t,t') + Q_p^*(t,t')Q_q(t,t')Q_r(t,t')G_k^*(t,t') \Big] \\ + \mathrm{c.c.}, \end{aligned} \tag{72}$$

where we have used that the integrand in (72) is anti-symmetric with respect to the interchange $\{\boldsymbol{k},\boldsymbol{p}\} \leftrightarrow \{\boldsymbol{q},\boldsymbol{r}\}$. In the particle-number-transfer range, $\Pi_{\rm n}(K)$ is a constant. Suppose we have $\Pi_{\rm n}(K) = \Pi_{\rm n}$ in the range $k_0 \ll K \ll k_1$.

The characteristic time scale $T_{\rm L}(k)$ associated to the linear term is same as (35). The characteristic time scale associated to $t-t'$ in $Q_k(t,t')$ and $G_k(t,t')$ in the particle-number-transfer range is in general different from $T_{\rm NL}(k)$ in the energy-transfer range and we denote it by $T_{\rm NL,n}(k)$.

Let us consider the ST region where $T_{\rm NL,n}(k) \ll T_{\rm L}(k)$. The factor $\mathrm{e}^{\frac{\mathrm{i}}{2m}(k^2+p^2-q^2-r^2)(t-t')}$ in (72) is approximated by 1 and we can assume that $Q_k(t,t')$ and $G_k(t,t')$ are real in the range. We assume the similarity forms,

$$Q_k = A k^a g^b |\Pi_{\rm n}|^c, \tag{73}$$

$$R_k(t,t') = \hat{R}(k^d g^e |\Pi_{\rm n}|^f (t-t')), \tag{74}$$

$$G_k(t,t') = \hat{G}(k^d g^e |\Pi_{\rm n}|^f (t-t')), \tag{75}$$

in the range and substitute them into (27), (28) and (72). Then, we find that

$$a=-3, \quad b=-\frac{1}{2}, \quad c=e=f=\frac{1}{2}, \quad d=0, \quad A=|I_3|^{-1/3}, \tag{76}$$

where

$$I_3 := \frac{64\pi^2}{(2\pi)^9}\int_0^1 \frac{\mathrm{d}\alpha}{\alpha}\int_0^\alpha \mathrm{d}q' \int_{|1-q'|}^{1+q'} \mathrm{d}l' \int_0^\infty \mathrm{d}p' \int_{|l'-p'|}^{l'+p'} \mathrm{d}r' p'q'r' \Big[(-1+q'^{-3})p'^{-3}r'^{-3}\Big] \times \int_0^\infty \mathrm{d}\tau \left[\hat{R}(\tau)\right]^3 \hat{G}(\tau), \tag{77}$$

and $\hat{R}$ and $\hat{G}$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{R}(\tau) = |I_3|^{-2/3}\frac{8\pi^2}{(2\pi)^6}\int_0^\infty \mathrm{d}q' \int_{|1-q'|}^{1+q'} \mathrm{d}l' \int_0^\infty \mathrm{d}p' \int_{|l'-p'|}^{l'+p'} p'q'r'p'^{-3}r'^{-3} \times \int_{-\infty}^{\tau} \mathrm{d}\tau' \left[\hat{R}(\tau-\tau')\right]^2 \left[-\hat{G}(\tau-\tau')\hat{R}(\tau') + q'^{-3}\hat{R}(\tau-\tau')\hat{G}(-\tau')\right], \tag{78}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{G}(\tau) = |I_3|^{-2/3}\frac{8\pi^2}{(2\pi)^6}\int_0^\infty \mathrm{d}q' \int_{|1-q'|}^{1+q'} \mathrm{d}l' \int_0^\infty \mathrm{d}p' \int_{|l'-p'|}^{l'+p'} p'q'r'p'^{-3}r'^{-3} \times \int_0^{\tau} \mathrm{d}\tau' \left[\hat{R}(\tau-\tau')\right]^2 \hat{G}(\tau-\tau')\hat{G}(\tau'). \tag{79}$$

(74), (75) and (76) imply that

$$T_{\rm NL,n}(k) = g^{-1/2}|\Pi_{\rm n}|^{-1/2}, \tag{80}$$

and therefore the wavenumber $k_{*,\rm n}$ at which $T_{\rm NL,n}(k) = T_{\rm L}(k)$ is given by

$$k_{*,\rm n} = (2m)^{1/2} g^{1/4} |\Pi_{\rm n}|^{1/4}. \tag{81}$$

From (73) and (76), we have

$$Q_k = |I_3|^{-1/3} g^{-1/2} |\Pi_{\rm n}|^{1/2} k^{-3}, \tag{82}$$

in the range $k_0 \ll k \ll \min(k_*, k_1)$.

The exponent $a = -3$ is marginal, that is, the wavenumber integrals in (77)–(79) are logarithmically divergent at the both ends of the integral range and the lower and higher cutoff wavenumber k_0 and $k_{\rm t} = \min(k_*, k_1)$, respectively, should be explicitly used in the analysis. The analysis would yield logarithmic correction of the form $Q_k \propto k^{-3}[\ln(k/k_0)]^{m_1}[\ln(k_{\rm t}/k)]^{m_2}$. Our preliminary analysis showed that $m_1 = m_2 = -1$ which implies that the divergence are not evaded and that the further correction is needed. We may expect I_3, with some cutoff introduced for the wavenumber integral, to be positive from the same reason as I_1. Then, $\Pi_{\rm n} > 0$, i.e., the particle number is transferred from low to high wavenumbers.

Let us consider the WWT region, $\max(k_{*,{\rm n}}, k_0) \ll k \ll k_1$. Since $T_{{\rm NL,n}}(k) \gg T_{\rm L}(k)$ in the range, $Q_k(t,t')$ and $G_k(t,t')$ in the integrand (72) are approximated by Q_k and 1, respectively. The approximated equation is equivalent to the corresponding equation of the WWT theory. By using (57) and substituting the similarity form

$$Q_k = A k^a (2m)^b g^c |\Pi_{\rm n}|^d \tag{83}$$

into (72), we find that

$$a = -\frac{7}{3}, \quad b = -\frac{1}{3}, \quad c = -\frac{2}{3}, \quad d = \frac{1}{3}, \quad A = |I_4|^{-1/3}, \tag{84}$$

with

$$I_4 = \frac{64\pi^3}{(2\pi)^9} \int_0^1 \frac{{\rm d}\alpha}{\alpha} \int_0^\alpha {\rm d}q' \int_{|1-q'|}^{1+q'} {\rm d}l' \int_0^\infty {\rm d}p' \int_{|l'-p'|}^{l'+p'} {\rm d}r' p' q' r' \\ \times \delta(1 + p'^2 - q'^2 - r'^2)(p'^a q'^a + p'^a r'^a - q'^a r'^a + p'^a q'^a r'^a). \tag{85}$$

(83), (84) and (85) imply that

$$Q_k = |I_4|^{-1/3} (2m)^{-1/3} g^{-2/3} |\Pi_{\rm n}|^{1/3} k^{-7/3}. \tag{86}$$

Obviously, (86) is same as the result from the WWT theory. It is known in the WWT theory that $I_4 < 0$ and the proof can be obtained in a similar way as in Appendix A. Therefore, we have $\Pi_{\rm n} < 0$, i.e., the particle number is transferred from high to low wavenumber.

When $k_0 \ll k_{*,{\rm n}} \ll k_1$, we have $\Pi_{\rm n} > 0$ for $k_0 \ll k \ll k_{*,{\rm n}}$ and $\Pi_{\rm n} < 0$ for $k_{*,{\rm n}} \ll k \ll k_1$. These imply accumulation of the particles around $k_{*,{\rm n}}$. The sink of particles around $k_{*,{\rm n}}$ is required in order to maintain the statistical stationarity. However, it may be difficult to think of physical situations in which the sink of particles is located around a particular wavenumber. When the sink is absent, the peak of $F(k)$ around $k_{*,{\rm n}}$ would increase with time and the (marginal) convergence of the integrals in the both ranges $k \ll k_{*,\mathrm{n}}$ and $k \gg k_{*,\mathrm{n}}$ would be violated at some time, which implies that (82) and (86) would not be sustained.

The results for the particle-number-transfer range are summarized in terms of one-dimensional spectrum $F(k)$ as follows. In the ST region, $k_0 \ll k \ll \min(k_{*,\mathrm{n}}, k_1)$,

$$F(k) = C_3 g^{-1/2} |\Pi_\mathrm{n}|^{1/2} k^{-1}, \tag{87}$$

with $C_3 := |I_3|^{-1/3}/(2\pi^2)$ and a possible logarithmic corrections or further corrections near the both ends of the wavenumber range. The direction of particle-number transfer is forward, i.e., $\Pi_\mathrm{n} > 0$. In the WWT region, $\max(k_{*,\mathrm{n}}, k_0) \ll k \ll k_1$,

$$F(k) = C_4 (2m)^{-1/3} g^{-2/3} |\Pi_\mathrm{n}|^{1/3} k^{-1/3}, \tag{88}$$

with $C_4 := |I_4|^{-1/3}/(2\pi^2)$. The direction of particle-number transfer is inverse, i.e., $\Pi_\mathrm{n} < 0$.

6. Discussion

In the present analysis, we assumed a constant energy flux or particle-number flux in a certain wavenumber range. For individual cases, whether which or both or none of the constant fluxes emerges depends on the manner of forcing and dissipation.

Let us assume that the forcing $B_{\boldsymbol{k}}(t)$ is applied in a narrow band range around k_B and $D_{\boldsymbol{k}}(t)$ is applied in $k < k_0$ and $k > k_1$ with $k_0 \ll k_B \ll k_1$. When k_B is located in the WWT region, i.e., $k_B \gg k_*, k_{*,\mathrm{n}}$, there will be a particle-number flux from k_B to the lower wavenumbers and a energy flux from k_B to the higher wavenumbers (see, for example, Ref. [12]). If the forcing is continuously applied and a statistically quasi-stationary state is achieved, the scaling law $F(k) \propto k^{-1/3}$ for the particle-number-transfer range in the WWT region will be observed in $\max(k_0, k_{*,\mathrm{n}}) \ll k \ll k_B$ and $F(k) \propto k^{-1}$ for the energy-transfer range in the WWT region will be observed in $k_B \ll k \ll k_1$. When $k_{*,\mathrm{n}} > k_0$, the particle-number-transfer range terminates at $k_{*,\mathrm{n}}$ and the particle accumulates around $k_{*,\mathrm{n}}$ since it is the positive particle-number flux that is allowed for the ST region $k < k_{*,\mathrm{n}}$.

When k_B is located in the ST region, i.e., $k_B \ll k_*, k_{*,\mathrm{n}}$, both the energy flux and the particle-number flux are positive and whether which of the transfer range emerges in the range $k \gg k_B$ would depend on more details of the forcing. When the constant energy flux is realized, $F(k) \propto k^{-2}$ for the energy-transfer range in the ST region will be observed in $k_B \ll k \ll \min(k_*, k_1)$ and $F(k) \propto k^{-1}$ for the energy-transfer range in the WWT region will be observed in $k_* \ll k \ll k_1$ when $k_* \ll k_1$.

The spectrum $F(k) \propto k^{-2}$ obtained in the wavenumber range $k > k_B$ of the numerical simulation of PNO is in support of the prediction of the closure analysis for the energy transfer range in the ST region. The spectrum $F(k) \propto k^{-2}$ is also obtained in our new numerical simulation without forcing and dissipation which is referred to in Sec. 1. This suggests that, when $F(k)$ is initially distributed in the ST region, it will freely evolve to form constant energy flux to the higher wavenumber.

In PNO, k^{-2} spectrum for large $\bar{n}$ is explained by a phenomenological "critical balance" proposition. Although the scaling k^{-2} is the same, their phenomenology is essentially different from the present closure result (68). In PNO phenomenology, it is assumed that the nonlinear interactions are local in the Fourier space and that two time scales $T_{\mathrm{L}}(k)$ and $T_{\mathrm{NL}}(k)$ are of the same order. On the contrary, in the present closure analysis, there are some dominant contributions from outside the inertial range in the wavevector integration, which implies the non-locality, and $T_{\mathrm{NL}}(k) \ll T_{\mathrm{L}}(k)$. Note that the non-locality and the time scale separation are not the assumptions but they are derived to be consistent with the similarity solution of the closure equations. The difference appears also in the resulting spectrum $F(k)$. When PNO phenomenology is applied to arbitrary spatial dimension d, $F(k)$ is given by

$$F(k) \propto L^d (2m)^{-1} g^{-1} k^{1-d}, \tag{89}$$

where L is the system size. PNO phenomenology explicitly depends on L and d. Although we showed explicit formulas for $d = 3$ in this paper, the closure results for $F(k)$ will be unchanged, except for the non-dimensional constants, for arbitrary d. Thus, the present closure result (68) does not depend on L and d. On the other hand, (68) depends on the energy flux Π whereas the spectrum from the PNO phenomenology is independent of Π. Whether the present closure analysis or PNO phenomenology is the correct explanation for the results from the numerical simulations would be checked by changing the parameters in the numerical simulations.

Appendix A. Proof of $I_2 > 0$

The proof is obtained by a similar method used in Refs. [9, 10]. From (63), two alternative expressions for $I_2(a)$ can be obtained by applying two types of change of variables $\{\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}\} \to \{(k/p)\boldsymbol{p}, (k/p)\boldsymbol{q}, (k/p)\boldsymbol{k}\}$ and $\{(k/q)\boldsymbol{q}, (k/q)\boldsymbol{k}, (k/q)\boldsymbol{p}\}$. Taking average over the original expression of I_2 (63) and the two alternative expressions, introducing $\{\boldsymbol{p}', \boldsymbol{q}'\} := \{\boldsymbol{p}/k, \boldsymbol{q}/k\}$ and performing integral over k yield

$$\begin{aligned}
I_2(a) = &-\frac{4\pi}{3}\frac{K^{6+2a}}{6+2a}\int \frac{\mathrm{d}\boldsymbol{e_k}}{(2\pi)^3}\int_{\boldsymbol{p}'\boldsymbol{q}'} (p'q')^a \\
&\times \Big[\delta_{\boldsymbol{e_k}-\boldsymbol{p}'-\boldsymbol{q}'}\delta(1 - p'^2 - q'^2)(-1 + p'^{-a} + q'^{-a})(1 - p'^{-4-2a} - q'^{-4-2a}) \\
&+ \delta_{\boldsymbol{e_k}-\boldsymbol{p}'+\boldsymbol{q}'}\delta(1 - p'^2 + q'^2)(-1 + p'^{-a} - q'^{-a})(1 - p'^{-4-2a} + q'^{-4-2a}) \\
&+ \delta_{\boldsymbol{e_k}+\boldsymbol{p}'-\boldsymbol{q}'}\delta(1 + p'^2 - q'^2)(-1 - p'^{-a} + q'^{-a})(1 + p'^{-4-2a} - q'^{-4-2a})\Big],
\end{aligned} \tag{A.1}$$

where $\boldsymbol{e_k} := \boldsymbol{k}/k$ and $\int \mathrm{d}\boldsymbol{e_k}$ denote the integral over the solid angle of $\boldsymbol{k}$. Both the numerator and the denominator in the r.h.s. of (A.1) approach 0 in the limit $a \to -3+0$ and the limiting value can be obtained by taking the ratio of the derivatives with respect to a of the numerator and the denominator. It can be easily proved that $\mathrm{sgn}(k^x + p^x - q^x) = \mathrm{sgn}(y - x)$ and $k^y \ln k + p^y \ln p - q^y \ln q < 0$ for $k^y + p^y - q^y = 0$ and $x, y > 0$. Using these identities, we find that $I_2 = \lim_{a\to-3+0} I_2(a) > 0$.

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