Anisotropic Velocity Correlation Spectrum at Small Scales in a Homogeneous Turbulent Shear Flow

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A simple theoretical analysis and direct numerical simulations on 512^3 grid points suggest that the velocity correlation spectrum tensor in the inertial subrange of homogeneous turbulent shear flow at high Reynolds number is given by a simple form that is an anisotropic function of the wave vector. The tensor is determined by the rate of the strain tensor of the mean flow, the rate of energy dissipation per unit mass, the wave vector, and two nondimensional constants.

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Since the 1941 Kolmogorov theory, it has been widely accepted that the statistics of turbulence at very high Reynolds numbers is universal and isotropic in the small scale limit. Effect of anisotropy induced by forcing and/or boundary conditions at large scales should be weaker at smaller scales. However, recent quantitative studies suggest that the anisotropy at small scales persists in experimental and numerical anisotropic turbulence at finite Reynolds numbers with finite scale range (see, e.g., [1,2] and the references cited therein). The small scale anisotropy in turbulence is a theoretical challenge. The detailed information on the anisotropic process of energy transfer, in particular that in the inertial subrange, should be useful in turbulence modeling for practical problems. The purpose of this Letter is to investigate theoretically and numerically the quantitative effects of the mean flow on the small scale anisotropy in the inertial subrange of turbulence.

In this paper, we consider the incompressible turbulent shear flow that obeys the Navier-Stokes equations and has a simple mean flow profile $\boldsymbol{U}$; which is given by the simplest but nontrivial one, i.e., by a linear function of the position vector $\boldsymbol{x}$ as

$$U_i(\boldsymbol{x}) = S_{ij}x_j\,, \tag{1}$$

where S_{ij} is a time-independent constant 3×3 matrix satisfying $S_{ii} = 0$. The summation convention is used for repeated indices. This study is restricted to the small scale anisotropy of the second order correlations of the fluctuating velocity field, since much remains unknown about them and they are important in practical applications.

The mean flow $\boldsymbol{U}$ given by (1) is compatible with the homogeneity of single-time statistics; i.e., the statistics remain homogeneous if they were so at some initial instant. The single-time two-point correlation of the fluctuating velocity at points $\boldsymbol{x}$ and $\boldsymbol{x}'$ then depends on $\boldsymbol{x}$ and $\boldsymbol{x}'$ only through the difference $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$. It is convenient to define the Fourier transform of this quantity as $Q_{ij}(\boldsymbol{k}, t) \equiv (2\pi)^{-3} \int d^3\boldsymbol{r}\, \langle u_i(\boldsymbol{x}, t)u_j(\boldsymbol{x} + \boldsymbol{r}, t)\rangle e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}$.

The equation of motion governing the fluctuating field $\boldsymbol{u}$ contains two types of terms: (i) those representing the direct coupling of $\boldsymbol{u}$ and the mean flow, which are bilinear in $\boldsymbol{U}$ and $\boldsymbol{u}$; and (ii) those representing the nonlinear coupling within the fluctuating field, which are quadratic in $\boldsymbol{u}$. The mean field $\boldsymbol{U}$ has a spatially small gradient and is temporally coherent, whereas $\boldsymbol{u}$ may have a spatially larger gradient and temporally shorter correlation. These can be taken into account and the relative importance of these couplings (i) and (ii) is evident when their characteristic time scales are compared. The time scale τ_S associated with the coupling (i) is of order $1/S$, where $S \equiv \max_{ij}|S_{ij}|$, and independent of the wave number k. However, the time scale τ_N of eddies with a length scale of ℓ that is associated with (ii) is of order ℓ/u_ℓ, where u_ℓ is the characteristic velocity of the eddies. If we assume Kolmogorov scaling $u_\ell \sim (\epsilon\ell)^{1/3}$, then $\tau_N \sim (\ell^2/\epsilon)^{1/3} \sim [1/(\epsilon k^2)]^{1/3}$, where ϵ is the energy dissipation rate per unit mass, and $k \sim 1/\ell$. Thus, τ_N is dependent on the wave number, unlike τ_S, and it is much smaller than τ_S at large wave numbers. This suggests that in the inertial subrange, where k is much larger than the characteristic wave number k_0 of energy containing eddies in the fluctuating velocity field, the nonlinear coupling between the eddies is more dominant than direct interactions with the mean flow.

The above consideration leads us to assume that there exists a wave number range with $k \gg k_0$ in fully developed turbulence at high Reynolds numbers such that the energy spectrum is approximated by

$$Q_{ij}(\boldsymbol{k}) = Q_{ij}^{(0)}(\boldsymbol{k}) + Q_{ij}^{(1)}(\boldsymbol{k})\,, \tag{2}$$

$$Q_{ij}^{(0)}(\boldsymbol{k}) = \frac{K_o}{4\pi}\,\epsilon^{2/3}k^{-11/3}P_{ij}(\boldsymbol{k})\,, \tag{3}$$

where $Q_{ij}^{(0)}$ is the isotropic Kolmogorov spectrum in the wave number range $k_0 \ll k \ll k_d$. Here, $P_{ij}(\boldsymbol{k}) = \delta_{ij} - \hat{k}_i\hat{k}_j$, $\hat{k}_i = k_i/k$, K_o is the Kolmogorov constant, and $k_d = \epsilon^{1/4}\nu^{-3/4}$ is the Kolmogorov wave number. The term $Q_{ij}^{(1)}$ represents the modification due to the existence of the mean flow, and is assumed to be small for small τ_N/τ_S, i.e., small $\delta(k) \equiv S/(\epsilon k^2)^{1/3}$.

Let us further assume that (**A1**) the two-point statistics in this range are reflection invariant so that $Q_{ij}(\boldsymbol{k}) = Q_{ij}(-\boldsymbol{k})$, which implies the symmetry condition $Q_{ij}(\boldsymbol{k}) = Q_{ji}(\boldsymbol{k})$;

(A2) for small $\delta(k)$, only the linear terms in $\delta(k)$ need to be retained.

The latter implies that $Q_{ij}^{(1)}$ is linear in $S_{\alpha\beta}$. Therefore, there exists an isotropic fourth order tensor $C_{ij\alpha\beta}$ such that

$$Q_{ij}^{(1)}(\boldsymbol{k}) = C_{ij\alpha\beta}(\boldsymbol{k}) S_{\alpha\beta}\,. \tag{4}$$

Since Q_{ij} and $Q_{ij}^{(0)}$ are divergence free and symmetric, $Q_{ij}^{(1)}$ must also satisfy these conditions. The tensor $C_{ij\alpha\beta}(\boldsymbol{k})$ must therefore satisfy the following constraints: (I) $k_i C_{ij\alpha\beta}(\boldsymbol{k}) = k_j C_{ij\alpha\beta}(\boldsymbol{k}) = 0$, (II) $C_{ij\alpha\beta}(\boldsymbol{k}) = C_{ji\alpha\beta}(\boldsymbol{k})$. Under these constraints, $Q_{ij}^{(1)}(\boldsymbol{k})$ may be written, without loss of generality, in the form of (4) with

$$C_{ij\alpha\beta}(\boldsymbol{k}) = a(k)[P_{i\alpha}(\boldsymbol{k})P_{j\beta}(\boldsymbol{k}) + P_{i\beta}(\boldsymbol{k})P_{j\alpha}(\boldsymbol{k})] + b(k)P_{ij}(\boldsymbol{k})\hat{k}_\alpha \hat{k}_\beta\,, \tag{5}$$

for any traceless tensor $S_{\alpha\beta}$. Since $C_{ij\alpha\beta}(\boldsymbol{k})$ is independent of $S_{\alpha\beta}$, we assume that C depends only on ϵ and $\boldsymbol{k}$ in the inertial subrange, following Kolmogorov. The dimensional analysis then gives

$$a(k) = A\epsilon^{1/3}k^{-13/3}, \qquad b(k) = B\epsilon^{1/3}k^{-13/3}, \tag{6}$$

where A and B are universal constants.

The anisotropic spectrum $Q_{ij}^{(1)}(\boldsymbol{k})$ is determined for any $S_{\alpha\beta}$ using (4) with (5). In particular, when $S_{\alpha\beta} = S\delta_{\alpha 1}\delta_{\beta 2}$, the anisotropic components $E_{12}(k)$ and $E_{ii}^{12}(k)$ are related to A and B according to

$$E_{12}(k) = \frac{4\pi}{15}(7A - B)\xi\,, \qquad \frac{1}{2}E_{ii}^{12}(k) = \frac{4\pi}{15}(-A + B)\xi\,, \tag{7}$$

where $\xi \equiv \epsilon^{1/3}k^{-7/3}S = \epsilon^{2/3}k^{-5/3}\delta(k)$, $E_{ij}(k) \equiv \sum_{p=k} Q_{ij}(\boldsymbol{p})$, $E_{ij}^{ab}(k) \equiv \sum_{p=k} \hat{p}_a \hat{p}_b Q_{ij}(\boldsymbol{p})$, and $\sum_{p=k}$ denotes the integral or the summation with respect to $\boldsymbol{p}$ over a spherical shell that satisfies $k - 1/2 < p \le k + 1/2$.

The turbulence has been assumed above to be almost stationary in the inertial subrange (not at the larger scales). It is not difficult to include the effects of small nonstationarities in the dissipation rate ϵ, $\tau_\epsilon \equiv \epsilon(t)/[d\epsilon(t)/dt]$. However, it does not contribute to the anisotropic spectra to be considered below and therefore it is neglected in the present analysis.

The $k^{-13/3}$ dependence of $Q^{(1)}$ is in agreement with previous studies based on dimensional analysis, including Lumley [3]. However, there have been few investigations of the tensorial dependence of $Q^{(1)}$ on the vector $\boldsymbol{k}$ compared to studies of the spectrum scaling. Leslie [4] and Yoshizawa [5] did examine the tensorial dependence of $Q^{(1)}$, but their formulations of $Q^{(1)}$ are different from (4) with (5). The quantity $Q_{ij}^{(1)}(\boldsymbol{k})$ that was obtained by Leslie violates the symmetry condition $Q_{ij}(\boldsymbol{k}) = Q_{ji}(\boldsymbol{k})$, and does not satisfy the solenoidal condition; the one by Yoshizawa is based on extra assumptions of Eulerian direct interaction approximation and scale separation, and the $b(k)$ term in (5) is missing. The form (4) with (5) is derived solely from assumptions (**A1**) and (**A2**) and is new to the authors' knowledge. The form (4) with (5) may be Fourier transformed with respect to $\boldsymbol{k}$ into real space representation which belongs to the $j = 2$ sector of the SO(3) decomposition [6]. The present analysis shows that the linear modification due to the existence of the mean shear is completely determined by two elements in the sector.

Direct numerical simulation (DNS) data of homogeneous turbulence with an imposed simple shear mean flow (1) with $S_{ij} = S\delta_{i1}\delta_{j2}$ was analyzed to test the predictions of (4) with (5) on small scale anisotropy. The number of grid points is 512^3. Prior to this study, we performed a DNS with 1024^3 grid points to simulate almost statistically stationary forced turbulence without a mean flow field, using periodic boundary conditions with periods of 2π in each of the three Cartesian coordinate directions [7]. The initial conditions for our study were generated by cutting off the higher wave number components of the 1024^3 DNS field. The characteristic parameters of the initial velocity field are listed in Table I.

Under the presence of a simple shear mean flow, the initially periodic field remains periodic in the coordinate system moving with the mean flow. The position vector $\boldsymbol{X}$ is related to the position vector $\boldsymbol{x}$ in the fixed Eulerian frame, according to $X_1 = x_1 - Stx_2$, $X_2 = x_2$, and $X_3 = x_3$. The wave vector $\boldsymbol{K}$ in the moving frame is related to the wave vector in the fixed frame according to $\boldsymbol{K} \cdot \boldsymbol{X} = \boldsymbol{k} \cdot \boldsymbol{x}$. The present DNS fields are simulated using a fourth order Runge-Kutta method to advance the time. An alias-free spectral method was used with the wave vector $\boldsymbol{K}$ and a fast Fourier transform, in which the maximum wave number $K_{\max}$ of the retained modes is 241. The retained wave vector space is isotropic in $\boldsymbol{K}$ space, but not in $\boldsymbol{k}$ space.

Two simulations were performed with different shear rates, $S = 0.5$ and 1.0, up to time $St = 2.0$. Although the simulation domain in wave vector space is deformed, all the modes whose wave numbers are smaller than 100 remain in the domain throughout the simulation time. The characteristic wave number $k_S = \epsilon^{-1/2}S^{3/2}$, defined as the wave number at which $\delta(k_S) = 1$, is 1.3–1.5 and 3.3–3.8 for the runs with $S = 0.5$ and $S = 1.0$, respectively. The direct coupling to the mean flow is therefore smaller than the nonlinear coupling between the eddies of the fluctuating field across almost the entire simulated wave number range.

TABLE I. The characteristic quantities of the initial velocity field. R_λ, Taylor microscale Reynolds number; E, total energy; ϵ, energy dissipation rate; L_0, integral length scale; λ, Taylor microscale; η, Kolmogorov microscale.

R_λ	E	ϵ	L_0	λ	η
284	0.500	7.12×10^{-2}	1.21	0.143	4.30×10^{-3}

The anisotropic energy spectrum components $-E_{12}(k)$ and $-(1/2)E_{ii}^{12}(k)$ obtained from the $S = 0.5$ DNS with time $St = 2.0$ are shown in Fig. 1. The isotropic component $(1/2)E_{ii}(k)$ calculated at the same time is also shown for comparison. Both the anisotropic components $E_{12}(k)$ and $E_{ii}^{12}(k)$ are negative for all but the highest wave numbers, where the artificial effects of the simulation domain deformation may have a direct influence on the spectra. The Reynolds stress $-\langle u_1 u_2 \rangle$ is the integration of $-E_{12}(k)$ with respect to k. The negative values of $E_{12}(k)$ imply the positive Reynolds stress. Since $(1/2)(d\langle u_i u_i \rangle/dt) = -S\langle u_1 u_2 \rangle > 0$, where the viscous term is neglected, the energy is supplied from the mean flow to the fluctuating velocities. The values of $E_{12}(k)$ and $E_{ii}^{12}(k)$ in Fig. 1 show an approximate $k^{-7/3}$ power-law dependence at $k \approx 10$, which is in agreement with (6). Similar results were also observed when $S = 1.0$ (figure omitted). The spectra in this range are quasistationary over the simulated time interval, except during the initial transient state. This implies that the anisotropy is almost stationary, unlike the rapid-distortion limit (the linear limit ignoring the nonlinear term) [8].

Figure 2 shows the anisotropic components $-E_{12}(k)$ and $-(1/2)E_{ii}^{12}(k)$ normalized by $\xi = \epsilon^{1/3}k^{-7/3}S$. The plot suggests that the normalized spectrum $E_{12}(k)/\xi$ may be approximated by a constant in the vicinity of $k = 10$ for both runs between $1.0 \le St \le 2.0$. Also, the normalized spectrum $(1/2)E_{ii}^{12}(k)/\xi$ may be approximated by a constant over the same region, although a slight systematic decay with time is observed. The mean values of the normalized spectra for $S = 0.5$ and $St = 2.0$ are computed by taking the average values over $4 \le k \le 16$. In this region, the energy spectrum $E(k)$ exhibits nearly the $k^{-5/3}$ power law, and the energy flux $\Pi(k)$ is nearly constant with relative deviation from ϵ less than 10%. This gives

$$E_{12}(k) = (-0.60 \pm 0.16)\xi\,, \qquad \frac{1}{2}E_{ii}^{12}(k) = (-0.20 \pm 0.04)\xi\,, \tag{8}$$

where the error is estimated from the variance. The coefficients A and B, estimated from (7) and (8), are

$$A = -0.16 \pm 0.03, \qquad B = -0.40 \pm 0.06. \tag{9}$$

From (2), (4), and (5), some of the components of $E_{mn}^{ij}(i, j, m, n = 1, 2, 3)$ are identically zero when $S_{\alpha\beta} = \delta_{\alpha 1}\delta_{\beta 2}$, due to the symmetry in $\boldsymbol{k}$ space. Among the remaining components, those that are zero for isotropic turbulence are expressed in terms of A and B as

$$\begin{aligned} E_{12}^{11}(k) = E_{12}^{22}(k) &= (4\pi/105)(13A - 3B)\xi\,, \\ E_{12}^{33}(k) &= (4\pi/105)(23A - B)\xi\,, \\ E_{11}^{12}(k) = E_{22}^{12}(k) &= (16\pi/105)(-2A + B)\xi\,, \\ E_{33}^{12}(k) &= (8\pi/105)(A + 3B)\xi\,. \end{aligned} \tag{10}$$

This theoretical prediction can be tested using DNS data, as shown in Fig. 3. There is good agreement between the predictions and DNS.

Experiments have been performed to investigate the one-dimensional cross spectrum $\tilde{E}_{12}(k_1)$, which satisfies

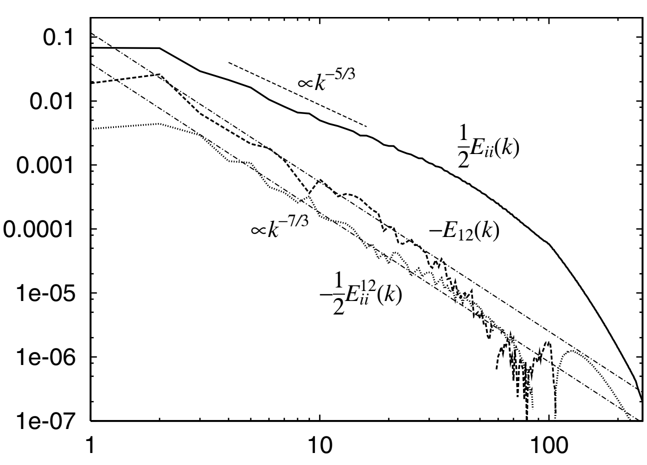

FIG. 1. The isotropic $(1/2)E_{ii}(k)$ (solid line), anisotropic $E_{12}(k)$ (dashed line), and $-(1/2)E_{ii}^{12}$ (dotted line) components of the energy spectrum at $S = 0.5$ and $St = 2.0$. The dot-dashed lines with a slope $-7/3$ are the line fits described by (8).

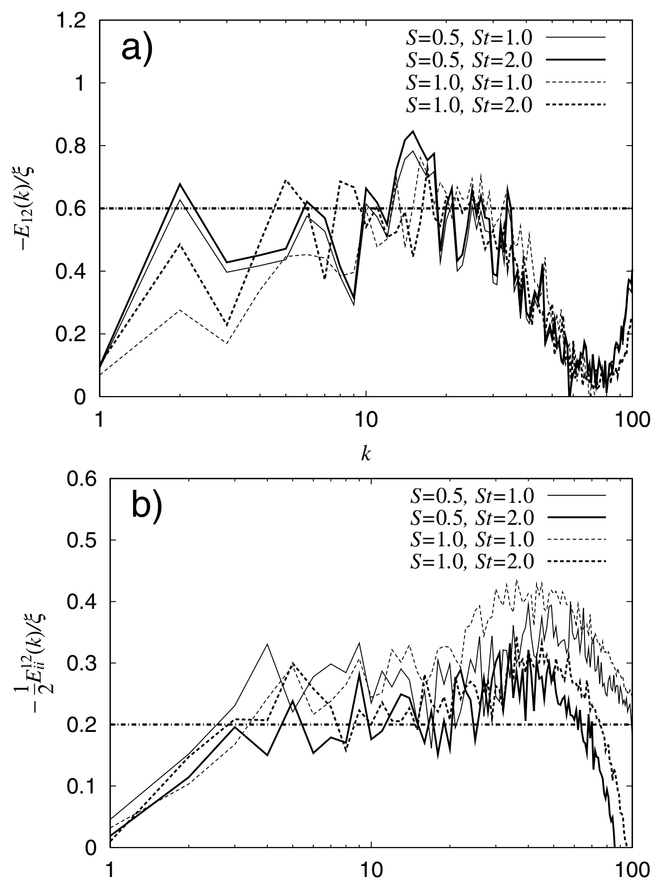

FIG. 2. The anisotropic components of the energy spectra (a) $-E_{12}(k)$ and (b) $-(1/2)E_{ii}^{12}(k)$ normalized by $\xi = \epsilon^{1/3}k^{-7/3}S$. The straight dot-dashed lines indicate the values given by (8).

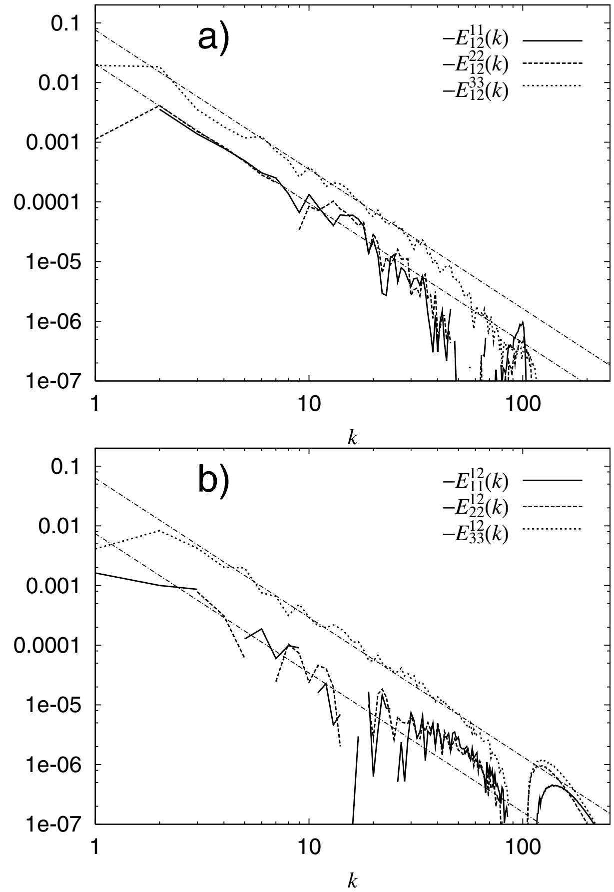

FIG. 3. Simulated tensor components, (a) $-E^{11}_{12}(k)$, $-E^{22}_{12}(k)$, $-E^{33}_{12}(k)$ and (b) $-E^{12}_{11}(k)$, $-E^{12}_{22}(k)$, $-E^{12}_{33}(k)$, at $S = 0.5$ and $St = 2.0$. Straight lines are the estimates using (9) and (10).

$\int_0^\infty dk_1\, \tilde{E}_{12}(k_1) = \langle u_1 u_2 \rangle$. From (2),(4)–(6), this can be expressed as $\tilde{E}_{12}(k_1) = -C_0 \xi_1$, $C_0 = (36\pi/1729) \times (-33A + 7B)$ in the scaling subrange, where $\xi_1 = \epsilon^{1/3} k_1^{-7/3} S$. Substituting the values of A and B in (9) to the above equation gives $C_0 = 0.16 \pm 0.05$. This is in fairly good agreement with experimental values ($\sim$0.14) obtained by Wyngaard and Cote [9] and by Saddoughi and Veeravalli [10]. On the other hand, in order to determine the universal constants A and B completely from experiments, measurement of another anisotropic spectrum which is linearly independent of $\tilde{E}_{12}(k_1)$ is needed.

In the present analysis, the linear assumption (A2) is used. The dimensional analysis used to derive (6) implicitly assumes that $C_{ij\alpha\beta}$ is independent of large scale flow statistics, except ϵ. These are not trivial assumptions. Recent studies of anomalous scaling of anisotropy show that the effect of the zero modes cannot be excluded *a priori* [11,12]. The agreement between the present theory and DNS suggests that the zero modes may not be significant in the problem under consideration.

The assumptions of our theory are consistent with spectral closure approximations such as the Lagrangian renormalized approximation (LRA) [13], which is free from any *ad hoc* adjusting parameter and yields the Kolmogorov spectrum for isotropic turbulence. The universal constants A and B may be estimated using the LRA by assuming the formulas (2)–(6) over the scaling range $k_0 < k < k_1$, and then substituting it into the closure equations. A preliminary analysis suggests that the estimate is sensitive to k_0/k for finite but small $k_0/k(\sim 0.2)$. Thus, the similarity range must be quite large to obtain a reliable estimate of the asymptotic values of A and B at high Reynolds numbers from experiments or DNS. The closure analysis based on the LRA is now underway, and the details will be reported elsewhere.

Since the gradient of the arbitrary mean flow is locally constant at sufficiently small scales, it is tempting to assume that the formulas (2)–(6) can be applied to general turbulent shear flows (which is at least formally possible). The validity of such an assumption is left to be examined in the future.

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