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7 Full-counting statistics and fluctuation theorem

This is the lecture note on Jun. 3, 2024 focusing on the application of the Full-counting statistics of the transferred electrons. For large time limit, we can obtain an explicit expression of the cumulant generation function and the noise behaviors for large and low bias limits are derived. Finally, it is shown that the cumulant generation function has special symmetry, which reduces to the *Fluctuation theorem*.

7.1 Summary of the last lecture

We are discussing the system of a single electron level coupled to two leads obeying following master equation

$$\frac{d}{dt}\left|W(t)\right\rangle = \hat{M}\left|W(t)\right\rangle. \tag{1}$$

We had introduced a modified transition matrix with a counting field χ, $\hat{M}(\chi)$. With this, we can define a characteristic function

$$Z_\tau(\chi) = \langle 0|e^{\hat{M}(\chi)\tau}|W(0)\rangle, \tag{2}$$

where τ is the measurement time. This provides a cumulant generating function of the transferred electrons,

$$F(\chi) = \lim_{\tau\to\infty}\frac{1}{\tau}\ln Z_\tau(\chi). \tag{3}$$

Using this cumulant generating function, we can evaluate the average current and the current noise,

$$I = -e\left.\frac{\partial F(\chi)}{\partial(i\chi)}\right|_{\chi=0}, \quad S = 2e^2\left.\frac{\partial^2 F(\chi)}{\partial(i\chi)^2}\right|_{\chi=0}. \tag{4}$$

Exercise V: Calculate the average current I and zero-frequency current noise S of a highly-biased quantum point contact and calculate its Fano factor. The detail of the assignment is uploaded in manaba. Report deadline is June 17, 13:00 to manaba.

7.2 Large τ behavior

We are interested in the probability distribution of the net transferred electron number n for large time τ. The eigen-equation of $\hat{M}(\chi)$, $(k = 0, 1)$ is

$$\hat{M}(\chi)\left|\psi_k(\chi)\right\rangle = \lambda_k(\chi)\left|\psi_k(\chi)\right\rangle. \tag{5}$$

The eigenvalues are determined by the relation $\left|\hat{M}(\chi) - \lambda(\chi)\hat{I}\right| = 0$, which provides

$$\lambda_{0,1}(\chi) = -\frac{\gamma_+ + \gamma_-}{2} \mp \sqrt{\left(\frac{\gamma_+ - \gamma_-}{2}\right)^2 + (\gamma_{L-} + \gamma_{R-}e^{i\chi})(\gamma_{L+} + \gamma_{R+}e^{-i\chi})}. \tag{6}$$

Choosing $\Re\lambda_0(\chi) > \Re\lambda_1(\chi)$ (note that both of which are negative) being satisfied for $|\chi| \ll 1$,

$$\lambda_0(\chi) = -\frac{\gamma_+ + \gamma_-}{2} + \sqrt{\left(\frac{\gamma_+ - \gamma_-}{2}\right)^2 + (\gamma_{L-} + \gamma_{R-}e^{i\chi})(\gamma_{L+} + \gamma_{R+}e^{-i\chi})}. \tag{7}$$

Clearly, $\lim_{\chi\to 0}\lambda_0(\chi) = 0$. For the left and right eigenfunctions, we require the orthonormality condition $\langle\psi_k(\chi)|\psi_q(\chi)\rangle = \delta_{kq}$ and completeness condition $\sum_{k=0,1}|\psi_k(\chi)\rangle\langle\psi_k(\chi)| = \hat{I}$.

Using the completeness relation, we expand the initial state $|W(0)\rangle$ with the right eigenvectors $|\psi_k(\chi)\rangle$,

$$|W(0)\rangle = \sum_{k=0,1} a_k(\chi)\,|\psi_k(\chi)\rangle\,, \tag{8}$$

where $a_k(\chi)$ $(= \langle\psi_k(\chi)|W(0)\rangle)$ are the expansion coefficients. Then we note that

$$1 = \langle 0|W(0)\rangle = \sum_{k=0,1} a_k(\chi)\,\langle 0|\psi_k(\chi)\rangle\,, \tag{9}$$

and

$$\lim_{\chi\to 0}\langle 0|\psi_k(\chi)\rangle = \delta_{k0}, \qquad \lim_{\chi\to 0} a_0(\chi) = 1. \tag{10}$$

Using this expansion, the characteristic function is

$$Z_\tau(\chi) = \left\langle 0 \left| e^{\hat{M}(\chi)\tau}\sum_{k=0,1} a_k(\chi)\right| \psi_j(\chi)\right\rangle = \sum_{k=0,1} a_k(\chi)\,\langle 0|e^{\lambda_k(\chi)\tau}|\psi_k(\chi)\rangle$$
$$\overset{\tau\to\infty}{\sim} a_0(\chi)\,\langle 0|\psi_0(\chi)\rangle\, e^{\lambda_0(\chi)\tau}, \tag{11}$$

where we noticed that the factor $e^{\lambda_1(\chi)\tau}$ decays much faster than $e^{\lambda_0(\chi)\tau}$ for $\tau\to\infty$. Hence, for large τ

$$\ln Z_\tau(\chi) \sim \ln\{a_0(\chi)\,\langle 0|\psi_0(\chi)\rangle\} + \tau\lambda_0(\chi) \sim \tau\lambda_0(\chi), \tag{12}$$

where the first term is progressively less dominant for large τ.

Using the results, we have

$$F(\chi) = \lim_{\tau\to\infty}\frac{1}{\tau}\ln Z_\tau(\chi) \sim \lambda_0(\chi). \tag{13}$$

Assuming local detail balance conditions for left and right leads, we defined following factors for $\nu = L/R$:

$$\gamma_\nu \equiv \gamma_{\nu+} + \gamma_{\nu-}, \tag{14}$$
$$\gamma_{\nu+} = \gamma_\nu f_\nu(\epsilon_0) \equiv \gamma_\nu f_\nu^+, \tag{15}$$
$$\gamma_{\nu-} = \gamma_\nu(1 - f_\nu(\epsilon_0)) \equiv \gamma_\nu f_\nu^-, \tag{16}$$

where we defined the Fermi distribution function of the lead ν,

$$f_\nu(\epsilon_0) \equiv \frac{1}{e^{\beta_\nu(\epsilon_0-\mu_\nu)}+1}. \tag{17}$$

With these, we have the expression of cumulant generation function:

$$F(\chi) = \lambda_0 = -\frac{\gamma_L + \gamma_R}{2} + \frac{\gamma_L + \gamma_R}{2}\sqrt{1 - u\left\{f_L^+ f_R^-(1 - e^{i\chi}) + f_L^- f_R^+(1 - e^{-i\chi})\right\}}, \tag{18}$$

where we defined $u \equiv 4\gamma_L\gamma_R/(\gamma_L + \gamma_R)^2$.

Using this result, the average current is

$$
\begin{aligned}
I &= -e\left.\frac{\partial F(\chi)}{\partial(i\chi)}\right|_{\chi=0}\\
&= -e\frac{\gamma_L+\gamma_R}{2}\left.\frac{-u\left\{f_L^+f_R^-(-e^{i\chi})+f_L^-f_R^+e^{-i\chi}\right\}}{2\sqrt{A}}\right|_{\chi=0}\\
&= -\frac{e(\gamma_L+\gamma_R)}{4}u\left\{f_L^+f_R^- - f_L^-f_R^+\right\}\\
&= -\frac{e\gamma_L\gamma_R}{\gamma_L+\gamma_R}\left\{f_L(\epsilon_0)(1-f_R(\epsilon_0))-(1-f_L(\epsilon_0))f_R(\epsilon_0)\right\}\\
&= -e\frac{\gamma_L\gamma_R}{\gamma_L+\gamma_R}(f_L(\epsilon_0)-f_R(\epsilon_0)),
\end{aligned}
\tag{19}
$$

where we set $A \equiv 1-u\left\{f_L^+f_R^-(1-e^{i\chi})+f_L^-f_R^+(1-e^{-i\chi})\right\}$. This expression is consistent with that obtained in the lecture note on May 13, Eq. (38).

Similarly, the current noise is

$$
\begin{aligned}
S &= 2e^2\left.\frac{\partial^2 F(\chi)}{\partial(i\chi)^2}\right|_{\chi=0}\\
&= 2e^2\frac{\partial}{\partial(i\chi)}\left.\left(\frac{\gamma_L+\gamma_R}{2}\frac{-u\left\{f_L^+f_R^-(-e^{i\chi})+f_L^-f_R^+e^{-i\chi}\right\}}{2\sqrt{A}}\right)\right|_{\chi=0}\\
&= 2e^2\frac{\gamma_L+\gamma_R}{4}\left.\left\{\frac{u(f_L^+f_R^-e^{i\chi}+f_L^-f_R^+e^{-i\chi})}{\sqrt{A}}-\frac{u^2\left(f_L^+f_R^-e^{i\chi}-f_R^+f_L^-e^{-i\chi}\right)^2}{2A^{3/2}}\right\}\right|_{\chi=0}\\
&= \frac{e^2(\gamma_L+\gamma_R)}{2}u\left[f_L^+f_R^-+f_L^-f_R^+-\frac{u}{2}\left(f_L^+f_R^--f_R^+f_L^-\right)^2\right]\\
&= 2e^2\frac{\gamma_L\gamma_R}{\gamma_L+\gamma_R}\left[f_L(\epsilon_0)+f_R(\epsilon_0)-2f_L(\epsilon_0)f_R(\epsilon_0)-\frac{2\gamma_L\gamma_R}{(\gamma_L+\gamma_R)^2}\left(f_L(\epsilon_0)-f_R(\epsilon_0)\right)^2\right].
\end{aligned}
\tag{20}
$$

We would like to inspect in detail the properties of the average current and current noise. In the following arguments, we assume the temperatures of two leads are the same, $\beta_L=\beta_R\equiv\beta=1/(k_\mathrm{B}T)$. We also define the bias, $eV\equiv\mu_L-\mu_R$, which is set to positive, hence the electron flows in average from the left to the right and the average current flows oppositely.

7.3 Large bias limit

Let us first discuss the large bias limit ($\mu_L-\epsilon_0, \epsilon_0-\mu_R \gg k_\mathrm{B}T$), then $f_L(\epsilon_0)\sim 1$ and $f_R(\epsilon_0)\sim 0$. The average current becomes

$$
I = -e\frac{\gamma_R\gamma_L}{\gamma_R+\gamma_L},
\tag{21}
$$

and the current noise becomes

$$
S = 2e^2\frac{\gamma_R\gamma_L}{\gamma_R+\gamma_L}\left(1-\frac{2\gamma_L\gamma_R}{(\gamma_L+\gamma_R)^2}\right).
\tag{22}
$$

Hence, the Fano factor is

$$
\frac{S}{2|I|} = e\left(1-\frac{2\gamma_L\gamma_R}{(\gamma_L+\gamma_R)^2}\right).
\tag{23}
$$

In the limit of $\gamma_L \ll \gamma_R$, $I\sim -e\gamma_L$, and

$$
\frac{S}{2|I|}\sim e\left(1-\frac{2\gamma_L/\gamma_R}{(1+\gamma_L/\gamma_R)^2}\right)\sim e,
\tag{24}
$$

which is the same result as the field emission (Poisson process) discussed in the lecture on May 20. In fact, cumulant generation function in this limit is

$$
\begin{aligned}
F(\chi) \sim \lambda_0(\chi) &\sim -\frac{\gamma_R}{2}\left(1+\frac{\gamma_L}{\gamma_R}\right) + \frac{\gamma_R}{2}\left(1+\frac{\gamma_L}{\gamma_R}\right)\sqrt{1-\frac{4\gamma_L/\gamma_R}{(1+\gamma_L/\gamma_R)^2}(1-e^{i\chi})} \\
&\sim -\frac{\gamma_R}{2}\left(1+\frac{\gamma_L}{\gamma_R}\right) + \frac{\gamma_R}{2}\left(1+\frac{\gamma_L}{\gamma_R}\right)\left\{1-\frac{2\gamma_L/\gamma_R}{(1+\gamma_L/\gamma_R)^2}(1-e^{i\chi})\right\} \\
&= -\gamma_L(1-e^{i\chi}),
\end{aligned} \tag{25}
$$

which is the cumulant generation function for the Poisson process as shown in the Appendix 1 of the last lecture note, May 27. Similar result is obtained for $\gamma_L \gg \gamma_R$, but in this case

$$
F(\chi) \sim -\gamma_R(1-e^{i\chi}). \tag{26}
$$

In contrast, when $\gamma_L = \gamma_R \equiv \gamma$, $I \sim -e\gamma/2$, and the Fano factor becomes

$$
\frac{S}{2|I|} \sim e\left(1-\frac{2\gamma^2}{(2\gamma)^2}\right) = \frac{e}{2}. \tag{27}
$$

The cumulant generation function is

$$
F(\chi) \sim -\gamma + \gamma\sqrt{1-\frac{4\gamma^2}{(2\gamma)^2}(1-e^{i\chi})} = -\gamma(1-e^{i\frac{\chi}{2}}), \tag{28}
$$

which is similar to the case $\gamma_L \ll \gamma_R$ but the exponential function with χ is different by a factor 1/2 and this cannot be assumed as a Poisson process. One should not interpret the result Eq. (27) as the elementary charge being half of the charge quantum since this is not a Poisson process. But this represents the "correlation effect" of the electron on the way through the quantum dots.

To see this correlation effect, let us calculate the number probability distribution function, $P(n,\tau)$, using the relation

$$
P(n,\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\chi Z_\tau(\chi) e^{-i\chi n}, \tag{29}
$$

whose derivation is given in Appendix A. Since we are focusing on the large bias limit, the contribution of $n<0$ can be neglected. With using the long-time ($\gamma\tau \gg 1$) form of the characteristic function

$$
Z_\tau(\chi) \sim e^{\tau F(\chi)}, \tag{30}
$$

we have

$$
P(n,\tau) \sim \frac{1}{2\pi}\int_{-\pi}^{\pi} d\chi e^{-\gamma\tau(1-e^{i\frac{\chi}{2}})-i\chi n}. \tag{31}
$$

We evaluate Eq. (31) in a saddle point approximation and we have

$$
P(n,\tau) \sim \sqrt{\frac{2}{\pi\gamma\tau}} e^{-\frac{2}{\gamma\tau}\left(\frac{\gamma\tau}{2}-n\right)^2}, \tag{32}
$$

for the derivation, see Appendix B. [1] From this distribution, we have

$$
\langle\hat{n}\rangle_\tau = \frac{\gamma\tau}{2}, \qquad \sigma_n^2 = \langle\hat{n}^2\rangle_\tau - \langle\hat{n}\rangle_\tau^2 = \frac{1}{2}\left(\frac{\gamma\tau}{2}\right), \tag{34}
$$

[1] If we apply the saddle point approximation to the case $\gamma_L \ll \gamma_R$, where the cumulant generation function is given by Eq. (25), we have the approximated number distribution function,

$$
P(n,\tau) \sim \sqrt{\frac{1}{2\pi\gamma_L\tau}} e^{-\frac{1}{2\gamma_L\tau}(\gamma_L\tau-n)^2}, \tag{33}
$$

which gives $\langle\hat{n}\rangle_\tau = \gamma_L\tau$ and $\sigma_n^2 = \gamma_L\tau$ obeying the characteristics of the Poisson distribution.

which has clearly narrower distribution than the that of the Poisson distribution, which should show $\langle \hat{n} \rangle_\tau = \sigma_n^2$. We numerically evaluate $P(n,\tau)$ for a fixed $\gamma\tau$ $(=40)$ as shown in Fig. 1. As a comparison, we also plot Poissonian distribution with a parameter $\gamma\tau/2$ which has the same average $\langle \hat{n} \rangle$. Obviously, the distribution of Eq. (31) to the left is narrower than that of the Poisson distribution shown to the right. Such a narrower distribution is called "*sub-Poissonian*" distribution, which means that the successive arrivals of the electrons are not completely random, but somewhat regularized compared with that of the Poisson case. In other words, the probability of the arrival of two successive electrons with very short separation is small, which is called as *anti-bunching effect*. This is reflected to the suppression of the current noise. Physically, this phenomena can be understood by the property of Fermi statistics (Pauli exclusion principle) such that no more than one electron can occupy the central quantum dot at the same time. Hence, the incoming electron from the left lead should wait until another electron occupying the quantum dot escapes to the right lead.

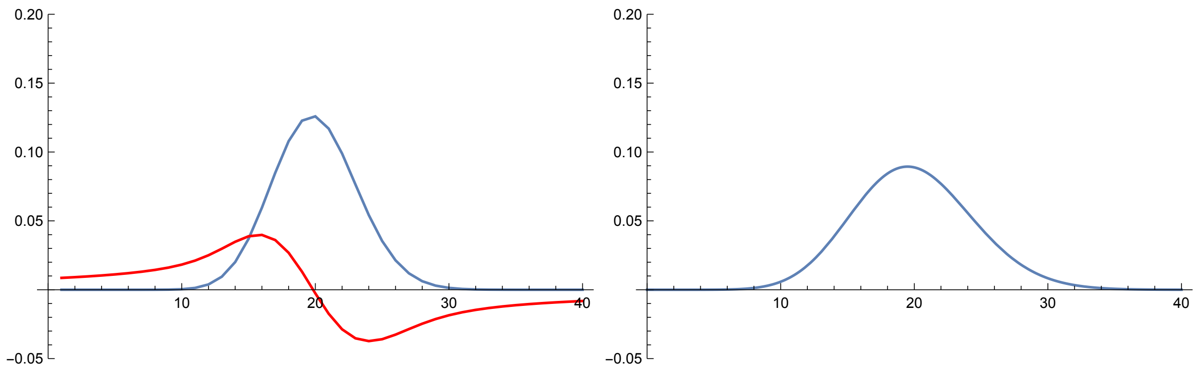

Figure 1: Left: Real part of the probability distribution function $P(n,\tau)$ (Eq. (31) in blue for $\gamma\tau = 40$ as a function of electron number n. Since the expression of (31) is an approximation, finite imaginary part (red solid line) remains. Right: Poissonian probability distribution function $P_{\rm Poisson}(n,\tau)$ for $\gamma\tau = 20$ (corresponding to the same average $\langle n \rangle$ with the left figure) as a function of n.

7.4 Low bias limit

Then, we consider the opposite limit, where $eV \ll k_{\rm B}T$. In the limit of low bias, we have

$$\begin{aligned} f_L(\epsilon_0) - f_R(\epsilon_0) &= f(\epsilon_0 - \mu_L) - f(\epsilon_0 - \mu_R) \\ &\sim - \left.\frac{\partial f(\epsilon)}{\partial \epsilon}\right|_{\epsilon=\epsilon_0-\mu_0} (\mu_L - \mu_R) \\ &= -f'(\epsilon_0 - \mu_0) eV, \end{aligned} \tag{35}$$

where we have introduced $f(\epsilon) \equiv (e^{\beta\epsilon}+1)^{-1}$, $f'(\epsilon) \equiv df(\epsilon)/(d\epsilon)$ and $\mu_0 \equiv (\mu_L + \mu_R)/2$. Therefore, the average current is

$$I \sim -e^2 \frac{\gamma_L\gamma_R}{\gamma_L+\gamma_R} \left\{-f'(\epsilon_0-\mu_0)\right\} V = -GV, \tag{36}$$

where we defined the *linear conductance*

$$G \equiv e^2 \frac{\gamma_L\gamma_R}{\gamma_L+\gamma_R} \left\{-f'(\epsilon_0-\mu_0)\right\}. \tag{37}$$

The current noise is

$$S \sim 4e^2 \frac{\gamma_L\gamma_R}{\gamma_L+\gamma_R} f(\epsilon_0-\mu_0)\left\{1 - f(\epsilon_0-\mu_0)\right\} + o(V^2). \tag{38}$$

Using the relation,

$$f'(\epsilon) = -\frac{\beta e^{\beta\epsilon}}{(e^{\beta\epsilon}+1)^2} = -\beta f(\epsilon)(1-f(\epsilon)), \tag{39}$$

the zero-bias noise is

$$S_{V=0} \sim 4e^2 \frac{\gamma_L \gamma_R}{\gamma_L + \gamma_R} \left(-\frac{1}{\beta} f'(\epsilon_0 - \mu_0) \right) = 4G k_{\mathrm{B}} T, \tag{40}$$

which is called the *Johnson-Nyquist noise.* Low bias noise has no physical significance more than the linear conductance.

7.5 Fluctuation theorm

The cumulant generation function derived in Eq. (18) satisfies following relation

$$F(\chi) = F(-\chi + iA), \tag{41}$$

where $A \equiv \beta(\mu_L - \mu_R)$ is called as an *affinity*, which characterizes the driving force of the non-equilibrium situation. This relation can be checked by noting

$$\begin{aligned} f_L^+ f_R^- (1 - e^{-i\chi - A}) + f_L^- f_R^+ (1 - e^{i\chi + A}) &= f_L^+ f_R^- - e^{-i\chi} f_L^+ f_R^- e^{-A} + f_L^- f_R^+ - e^{i\chi} f_L^- f_R^+ e^{A} \\ &= f_L^+ f_R^- (1 - e^{i\chi}) + f_L^- f_R^+ (1 - e^{-i\chi}), \end{aligned} \tag{42}$$

where we used the relation $f_\nu^- / f_\nu^+ = e^{\beta(\epsilon_0 - \mu_\nu)}$.

As shown in Eq. (29), the probability of n electrons transferred in a period τ is

$$P(n, \tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\chi Z_\tau(\chi) e^{-in\chi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\chi e^{\tau F(\chi)} e^{-in\chi}. \tag{43}$$

By using the relation Eq. (41),

$$\begin{aligned} P(n, \tau) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\chi e^{\tau F(-\chi + iA)} e^{-in\chi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\chi' e^{\tau F(\chi')} e^{-in(-\chi' + iA)} \\ &= e^{nA} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\chi' e^{\tau F(\chi')} e^{-i(-n)\chi'} = e^{nA} P(-n, \tau), \end{aligned} \tag{44}$$

where we changed the integration variable from χ to $\chi' = -\chi + iA$. For the bias condition $\mu_L > \mu_R$ $(A > 0)$ and for $n > 0$, this relation shows that the probability of transferring electrons in the reverse direction to the bias $n < 0$ is exponentially smaller than that of transferring in the forward direction. Defining the *entropy production* by transferring n electrons as $n\beta(\mu_L - \mu_R) \equiv \sigma$, this relation is rewritten as

$$\frac{P(\sigma)}{P(-\sigma)} = e^{\sigma}, \tag{45}$$

which is called the *Fluctuation Theorem.*

We can also obtain the relation

$$\begin{aligned} \langle e^{-\sigma} \rangle &= \sum_{n=-\infty}^{n} e^{-n\beta(\mu_L - \mu_R)} P(n, \tau) \\ &= \sum_{n=-\infty}^{n} P(-n, \tau) = 1, \end{aligned} \tag{46}$$

which is called the *Jarzynski equation.* For a convex function, $f''(x) \geq 0$ and probability distribution p_n, following *Jensen's relation* holds:

$$\sum_n p_n f(x_n) \geq f\left(\sum_n p_n x_n \right). \tag{47}$$

Using this relation, we have

$$1 = \langle e^{-\sigma} \rangle \geq e^{-\langle \sigma \rangle}, \tag{48}$$

we arrive the relation

$$\langle \sigma \rangle \geq 0, \tag{49}$$

which is the 2nd-law of the thermodynamics.

7.6 Conclusion

We have studied the full-counting statistics of an electron transport through a quantum dot determined by a Markovian master equation. When the tunneling rates are highly asymmetric, the distribution function of the transferred electron numbers in a time τ becomes a Poissonian for the large time limit. While for symmetric rates, the distribution is deviated from the Poissonian, and in fact becomes sub-Poissonian. For low bias, the zero-bias noise is the Johnson-Nyquist noise. Finally, the cumulant generation function is shown to obey the Fluctuation theorem and the entropy production satisfies the Jarzynski equation.

A Relation between $P(n, \tau)$ and $Z_\tau(\chi)$

As shown in the footnote in page 3 of the lecture note of May 27, the expectation value of a function of transferred numbers $\hat{n}$ of electrons in a time period τ, $\hat{O}(\hat{n})$, is given by

$$\langle \hat{O}(\hat{n}) \rangle_\tau \equiv \sum_{n=0}^{\infty} P(n, \tau) \hat{O}(n), \tag{50}$$

where $P(n, \tau)$ is the probability distribution function. If we choose a function $\hat{O}(\hat{n}) = \delta_{\hat{n}\ell}$ with some integer ℓ, we have

$$\langle \delta_{\hat{n}\ell} \rangle_\tau \equiv \sum_{n=0}^{\infty} P(n, \tau) \delta_{n\ell} = P(\ell, \tau). \tag{51}$$

The electron number generating function, $Z_\tau(\chi)$, provides m-th moment by

$$\left. \frac{\partial^m Z_\tau(\chi)}{\partial(i\chi)^m} \right|_{\chi=0} = \langle \hat{n}^m \rangle_\tau . \tag{52}$$

This relation suggests following series expansion of $Z_\tau(\chi)$,

$$\begin{aligned} Z_\tau(\chi) &= 1 + \langle \hat{n} \rangle_\tau (i\chi) + \frac{1}{2!} \langle \hat{n}^2 \rangle_\tau (i\chi)^2 + \frac{1}{3!} \langle \hat{n}^3 \rangle_\tau (i\chi)^3 + \cdots \\ &= \langle e^{i\chi\hat{n}} \rangle_\tau = \sum_{n=0}^{\infty} P(n, \tau) e^{i\chi n}. \end{aligned} \tag{53}$$

This relation can be interpreted as a (discrete) Fourier transform of the function $P(n, \tau)$. Then, let us evaluate an inverse Fourier transform of $Z_\tau(\chi)$ with some integer ℓ,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\chi Z_\tau(\chi) e^{-i\chi\ell} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\chi \langle e^{i\chi\hat{n}} \rangle_\tau e^{-i\chi\ell} \\ &= \left\langle \frac{1}{2\pi} \int_{-\pi}^{\pi} d\chi e^{i\chi(\hat{n}-\ell)} \right\rangle_\tau = \langle \delta_{\hat{n}\ell} \rangle = P(\ell, \tau), \end{aligned} \tag{54}$$

where we had used the relation valid for an integer ℓ, $\frac{1}{2\pi} \int_{-\pi}^{\pi} d\chi e^{i\chi\ell} = \delta_\ell$.

B Saddle point approximation

Here, we evaluate the integral

$$P(n, \tau) \sim \frac{1}{2\pi} \int_{-\pi}^{\pi} d\chi e^{-\gamma\tau(1-e^{i\frac{\chi}{2}})-i\chi n}, \tag{55}$$

under the long-time condition $\gamma\tau \gg 1$. The real part of the exponent of the integration kernel, $-\gamma\tau(1 - \cos\frac{\chi}{2})$, is zero at $\chi = 0$ and has very large negative values for other values of χ. Since $\gamma\tau \gg 1$, the integral is dominated by the values of $\chi \sim 0$. Hence, by expanding the exponent around $\chi = 0$ and neglecting the terms more than the second order, we have

$$-\gamma\tau(1-e^{i\frac{\chi}{2}}) - i\chi n \sim -\frac{\gamma\tau}{8}\chi^2 + i\frac{\gamma\tau}{2}\chi - in\chi = -\frac{\gamma\tau}{8}\left\{\chi - \frac{4i}{\gamma\tau}\left(\frac{\gamma\tau}{2}-n\right)^2\right\}^2 - \frac{2}{\gamma\tau}\left(\frac{\gamma\tau}{2}-n\right)^2, \quad (56)$$

where in the last equation, we completed the square of χ. By extending the integration region of Eq. (55) to $[-\infty, \infty]$ and executing the Gaussian integral over χ, we have the expression

$$P(n,\tau) \sim \sqrt{\frac{2}{\pi\gamma\tau}} e^{-\frac{2}{\gamma\tau}\left(\frac{\gamma\tau}{2}-n\right)^2}. \quad (57)$$

This probability distribution has a proper normalization,

$$\begin{aligned}\sum_{n=0}^{\infty} P(n,\tau) &= \sqrt{\frac{2}{\pi\gamma\tau}} \sum_{n=0}^{\infty} e^{-\frac{2}{\gamma\tau}\left(\frac{\gamma\tau}{2}-n\right)^2} \\ &\sim \sqrt{\frac{2}{\pi\gamma\tau}} \sum_{n=-\infty}^{\infty} e^{-\frac{2}{\gamma\tau}\left(\frac{\gamma\tau}{2}-n\right)^2} \\ &\sim \sqrt{\frac{2}{\pi\gamma\tau}} \int_{-\infty}^{\infty} dx e^{-\frac{2}{\gamma\tau}x^2} = 1, \end{aligned} \quad (58)$$

where in the last part we changed the variable $x = \frac{\gamma\tau}{2} - n$ and modified the sum into an integral.

References

[1] 齊藤圭司、「非平衡輸送現象：輸送現象における計数統計を学ぶための基礎（講義ノート）」、物性研究 **92**, 345-376 (2009).

[2] 内海裕洋、「量子ドットにおける完全計数統計」固体物理 **41**, 909 (2006).