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6 Full-counting statistics - classical

This is the lecture note on May 27, 2024 focusing on the average current and noise observed in a single-level quantum dot coupled to two leads, which is obeying a classical master equation. In contrast to the discussions in the previous lecture, we consider the electron motion in both (left to right/right to left) directions. The notion of full-counting statistics is introduced.

6.1 Master equation

First we review our discussion on the classical master equation. We consider stochastic processes of electrons in a quantum dot made of single level coupled with two leads, and disregard the spin degree of freedom. Hence, possible states in a quantum dot is either empty "0" or filled "1". We define the probabilities of these states at time t as $W_0(t)$ and $W_1(t)$, respectively, which satisfy $W_0(t) + W_1(t) = 1$ for all $t \geq 0$. We introduce the rate of an electron tunneling from the left (right) lead to the quantum dot as γ_{L+} (γ_{R+}) and the rate from the quantum dot to the left (right) lead as γ_{L-} (γ_{R-}). From the physical argument $\gamma_{L\pm}, \gamma_{R\pm} > 0$. Assuming the leads being in local equilibria, local detail balance condition reads

$$\frac{\gamma_{L+}}{\gamma_{L-}} = e^{-\beta_L(\epsilon_0-\mu_L)}, \quad \frac{\gamma_{R+}}{\gamma_{R-}} = e^{-\beta_R(\epsilon_0-\mu_R)}, \tag{1}$$

where ϵ_0 is the energy level of the quantum dot and $\beta_{L/R}, \mu_{L/R}$ are the inverse temperatures and chemical potentials of each lead, respectively.

We represent a "state" of the quantum dot at time t using Dirac's notation

$$|W(t)\rangle \equiv \begin{pmatrix} W_0(t) \\ W_1(t) \end{pmatrix}, \tag{2}$$

which is in fact a column vector and the master equation is

$$\frac{d}{dt}|W(t)\rangle = \hat{M}|W(t)\rangle, \quad \hat{M} = \begin{pmatrix} -\gamma_+ & \gamma_- \\ \gamma_+ & -\gamma_- \end{pmatrix}, \tag{3}$$

where $\hat{M}$ is the transition matrix. Formal solution of the master equation is

$$|W(t)\rangle = e^{\hat{M}t}|W(0)\rangle, \tag{4}$$

where $|W(0)\rangle$ is the initial ($t=0$) probability distribution. The time-development is Markovian, namely, for $0 < t' < t$,

$$|W(t)\rangle = e^{\hat{M}(t-t'+t')}|W(0)\rangle = e^{\hat{M}(t-t')}e^{\hat{M}t'}|W(0)\rangle = e^{\hat{M}(t-t')}|W(t')\rangle. \tag{5}$$

We introduce a special row vector

$$\langle 0| \equiv (1,1), \tag{6}$$

which provides the norm of the state (at all t)

$$\langle 0|W(t)\rangle = W_0(t) + W_1(t) = 1. \tag{7}$$

We can also define various "average" of a physical quantity $\hat{O}$ at time t by

$$\langle \hat{O} \rangle (t) \equiv \langle 0|\hat{O}|W(t)\rangle \,, \tag{8}$$

where $\hat{O}$ is a two-by-two matrix representing the physical quantity of the quantum dot. For example, the average number of electrons occupying the quantum dot, $\hat{N}$, is

$$\hat{N} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{9}$$

where $\langle \hat{N} \rangle (t) = \langle 0|\hat{N}|W(t)\rangle = W_1(t)$.

The eigenvalues λ_k $(k = 0, 1)$ and corresponding left/right eigenvectors of $\hat{M}$ are defined as

$$\langle k| \hat{M} = \lambda_k \langle k| \,, \quad \hat{M} |k\rangle = \lambda_k |k\rangle \,, \tag{10}$$

where λ_k are in general complex numbers and the left and right eigenvectors are $\{\langle k|\}^\dagger \neq |k\rangle$ since the matrix $\hat{M}$ is not Hermite. These also satisfy orthonormalization condition $\langle k|\ell\rangle = \delta_{k\ell}$. In particular, $k = 0$ corresponds the steady state with $\lambda_0 = 0$ and $|0\rangle \equiv |W_{\rm st}\rangle$. Since the property $\langle 0|e^{\hat{M}t}|W(0)\rangle = \langle 0|W(0)\rangle = 1$ holds for any initial probability distribution $|W(0)\rangle$, we have the relation $\langle 0| e^{\hat{M}t} = \langle 0|$.

We consider stochastic processes from $t = 0$ to $t = \tau$ (> 0). We discretize the time period $[0, \tau]$ into N $(\gg 1)$ small segments $\Delta\tau \equiv \tau/N$ and name the N-times in-between as $t_i \equiv i\tau/N$ $(i = 1, \dots, N)$. Set S_i $(= 0, 1)$ is the number of electrons in the quantum dot at time t_i, which is actually a stochastic random variable. We introduce a conditional distribution function

$$P(S_N, S_{N-1}, \cdots, S_{i+1}|S_i), \tag{11}$$

which is the conditional probability of series of charge states $\{S_N, S_{N-1}, \cdots, S_{i+1}\}$ conditioned with S_i at time t_i. Assuming Markov process, the relation

$$P(S_N, S_{N-1}, \cdots, S_j, S_{j-1}, \cdots, S_{i+1}|S_i) = P(S_N, S_{N-1}, \cdots, S_j|S_{j-1}) P(S_{j-1}, \dots, S_{i+1}|S_i), \tag{12}$$

holds. The conditional probabilities from t_i to $t_{i+1} = t_i + \Delta\tau$, $P(S_{i+1}|S_i)$, are explicitly given by

$$P(0|0) = 1 - \gamma_+ \Delta\tau, \;\; P(0|1) = \gamma_- \Delta\tau, \tag{13}$$

$$P(1|0) = \gamma_+ \Delta\tau, \;\; P(1|1) = 1 - \gamma_- \Delta\tau, \tag{14}$$

which satisfy the required relation $P(0|0) + P(1|0) = 1$ and $P(0|1) + P(1|1) = 1$. In the limit $N \to \infty$, neglecting the higher order terms in $\Delta\tau$,

$$e^{\hat{M}\Delta\tau} \sim \hat{I} + \hat{M}\Delta\tau = \begin{pmatrix} P(0|0) & P(0|1) \\ P(1|0) & P(1|1) \end{pmatrix}. \tag{15}$$

From these definitions, the probability that the quantum dot is empty at time $t = t_i$ is

$$W_0(t_i) = \sum_{S_{i-1}, S_{i-2}, \cdots, S_1, S_0} P(0, S_{i-1}, S_{i-2}, \cdots, S_1|S_0) W_{S_0}(0) = \sum_{S_{i-1}=0,1} P(0|S_{i-1}) W_{s_{i-1}}(t_{i-1}), \tag{16}$$

and the probability that the quantum dot is filled at time $t = t_i$ is

$$W_1(t_i) = \sum_{S_{i-1}, S_{i-2}, \cdots, S_1, S_0} P(1, S_{i-1}, S_{i-2}, \cdots, S_1|S_0) W_{S_0}(0) = \sum_{S_{i-1}=0,1} P(1|S_{i-1}) W_{s_{i-1}}(t_{i-1}). \tag{17}$$

In the Dirac notation and using Eq. (15), we have

$$\begin{aligned} |W(t_i)\rangle &= \sum_{S_{i-1}=0,1} \begin{pmatrix} P(0|S_{i-1}) W_{S_{i-1}}(t_{i-1}) \\ P(1|S_{i-1}) W_{S_{i-1}}(t_{i-1}) \end{pmatrix} = \begin{pmatrix} P(0|0) & P(0|1) \\ P(1|0) & P(1|1) \end{pmatrix} \begin{pmatrix} W_0(t_{i-1}) \\ W_1(t_{i-1}) \end{pmatrix} \\ &\sim e^{\hat{M}\Delta\tau} |W(t_{i-1})\rangle = e^{\hat{M}\Delta\tau} e^{\hat{M}\Delta\tau} |W(t_{i-2})\rangle = \cdots = e^{\hat{M}t_i} |W(0)\rangle \,, \end{aligned} \tag{18}$$

which is identical with Eq. (4).

6.2 Counting statistics

We introduce a "current" at time t_i, $I(t_i)$, where $\frac{\Delta\tau}{-e}I(t_i)$ is the net electron number (the difference of numbers of electrons moved to the right and that to the left) moved from the dot to the right lead in the period $[t_{i-1}, t_i]$. (The current between the left lead and the dot can be evaluated similarly, but here we focus on the value mentioned above.) Then the total number of moved electrons, n, in a time period τ is

$$n = \frac{\Delta\tau}{-e}\sum_{i=1}^{N} I(t_i), \tag{19}$$

which is a random number (could be a positive or negative integer). The average of n^m, m-th moment

$$\langle \hat{n}^m \rangle_\tau = \left(\frac{\Delta\tau}{-e}\right)^m \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_m=1}^{N} \langle I(t_{i_1})I(t_{i_2})\cdots I(t_{i_{m-1}})I(t_{i_m})\rangle \tag{20}$$

is related to the correlation of currents at m different times. [1] For example, we first evaluate the average of the current at $t = t_i$. Noticing the direction of the motion of the electron and that of the current are opposite, the current is finite (positive) when the dot is empty at $t = t_{i-1}$ and transition occurs with the rate γ_{R+} and is negative when the dot is filled at $t = t_{i-1}$ and transition occurs with the rate γ_{R-}, hence

$$\begin{aligned}
\langle I(t_i)\rangle &= (-e)\sum_{S_{i-2},S_{i-3},\cdots,S_1,S_0} \left\{-\gamma_{R+}P(0, S_{i-2},\cdots,S_1|S_0) + \gamma_{R-}P(1, S_{i-2},\cdots,S_1|S_0)\right\} W_{S_0}(0) \\
&= (-e)\left\{-\gamma_{R+}W_0(t_{i-1}) + \gamma_{R-}W_1(t_{i-1})\right\}.
\end{aligned} \tag{22}$$

The correlation of the currents at $t = t_i$ and $t = t_j$ can be evaluated similarly. Assuming $t_i > t_j$ and noting there are in total four processes of different signs that contribute to the product of the currents,

$$\begin{aligned}
\langle I(t_i)I(t_j)\rangle = e^2 &\sum_{S_{i-2},\cdots,S_{j+1}}\ \sum_{S_{j-2},\cdots S_1,S_0} \\
&\Big\{-\gamma_{R+}P(0, S_{i-2}, S_{i-3},\cdots,S_{j+1}|1)(-\gamma_{R+})P(0, S_{j-2},\cdots,S_1|S_0) \\
&- \gamma_{R+}P(0, S_{i-2}, S_{i-3},\cdots,S_{j+1}|0)\gamma_{R-}P(1, S_{j-2},\cdots,S_1|S_0) \\
&+ \gamma_{R-}P(1, S_{i-2}, S_{i-3},\cdots,S_{j+1}|1)(-\gamma_{R+})P(0, S_{j-2},\cdots,S_1|S_0) \\
&+ \gamma_{R-}P(1, S_{i-2}, S_{i-3},\cdots,S_{j+1}|0)\gamma_{R-}P(1, S_{j-2},\cdots,S_1|S_0)\Big\} W_{S_0}(0) \\
&= e^2 \sum_{S_{i-2},\cdots,S_{j+1}} \\
&\Big\{\gamma_{R+}P(0, S_{i-2}, S_{i-3},\cdots,S_{j+1}|1)\gamma_{R+}W_0(t_{j-1}) - \gamma_{R+}P(0, S_{i-2}, S_{i-3},\cdots,S_{j+1}|0)\gamma_{R-}W_1(t_{j-1}) \\
&- \gamma_{R-}P(1, S_{i-2}, S_{i-3},\cdots,S_{j+1}|1)\gamma_{R+}W_0(t_{j-1}) + \gamma_{R-}P(1, S_{i-2}, S_{i-3},\cdots,S_{j+1}|0)\gamma_{R-}W_1(t_{j-1})\Big\}.
\end{aligned} \tag{23}$$

In principle, we can calculate any correlations of currents appearing in (20) but in practice it is quite cumbersome. Fortunately, there is an ingenious way to calculate any order of correlations. Although it seems top-down, we introduce a transition matrix with a *counting field*, χ,

$$\hat{M}(\chi) = \begin{pmatrix} -\gamma_+ & \gamma_{L-} + \gamma_{R-}e^{i\chi} \\ \gamma_{L+} + \gamma_{R+}e^{-i\chi} & -\gamma_- \end{pmatrix}, \tag{24}$$

[1] Here, the formal meaning of $\langle\cdots\rangle_\tau$ for some physical quantity $\hat{O}(n)$ is

$$\langle \hat{O}(n)\rangle_\tau = \sum_{n=0}^{\infty} P(n,\tau)\hat{O}(n), \tag{21}$$

where $P(n,\tau)$ is the probability distribution of the random number n in the period τ. This physically means the ensemble average of the measurement for a period τ many times.

with $\hat{M}(\chi=0)=\hat{M}$. Now, an operator for the current is defined as

$$\mathcal{J}_R = -e\left.\frac{\partial \hat{M}(\chi)}{\partial(i\chi)}\right|_{\chi=0} = -e\begin{pmatrix} 0 & \gamma_{R-} \\ -\gamma_{R+} & 0 \end{pmatrix}. \tag{25}$$

Then the average of the current, Eq. (22), is

$$\begin{aligned}\langle I(t_i)\rangle &= (-e)\left\{-\gamma_{R+}W_0(t_{i-1})+\gamma_{R-}W_1(t_{i-1})\right\} \\ &= \langle 0|\,(-e)\begin{pmatrix} 0 & \gamma_{R-} \\ -\gamma_{R+} & 0 \end{pmatrix}|W(t_{i-1})\rangle \\ &= \langle 0|e^{\hat{M}(0)(\tau-t_i)}\mathcal{J}_R e^{\hat{M}(0)t_{i-1}}|W(0)\rangle\,,\end{aligned} \tag{26}$$

where we used the relation $\langle 0|\,e^{\hat{M}(0)t}=\langle 0|$. The current correlation for $t_i>t_j$ is similarly given by

$$\langle I(t_i)I(t_j)\rangle = \langle 0|e^{\hat{M}(0)(\tau-t_i)}\mathcal{J}_R e^{\hat{M}(0)(t_{i-1}-t_j)}\mathcal{J}_R e^{\hat{M}(0)t_{j-1}}|W(0)\rangle\,. \tag{27}$$

To proceed further, we introduce a *characteristic function*

$$\begin{aligned}Z_\tau(\chi) &= \lim_{N\to\infty}\langle 0|e^{\hat{M}(\chi)\Delta\tau}e^{\hat{M}(\chi)\Delta\tau}\cdots e^{\hat{M}(\chi)\Delta\tau}|W(0)\rangle \\ &= \lim_{N\to\infty}\langle 0|e^{\hat{M}(\chi)\tau}|W(0)\rangle\,.\end{aligned} \tag{28}$$

Then the first moment of n, defined in Eq. (19) with $N\to\infty$ is

$$\begin{aligned}\langle \hat{n}\rangle_\tau &= \left\langle \int_0^\tau \frac{dt}{-e}I(t)\right\rangle \\ &= \int_0^\tau \frac{dt}{-e}\,\langle 0|e^{\hat{M}(0)(\tau-t)}\mathcal{J}_R e^{\hat{M}(0)t}|W(0)\rangle \\ &= \lim_{N\to\infty}\sum_{i=1}^{N}\frac{\Delta\tau}{-e}\,\langle 0|e^{\hat{M}(0)(\tau-t_i)}\mathcal{J}_R e^{\hat{M}(0)\Delta\tau}e^{\hat{M}(0)t_{i-1}}|W(0)\rangle \\ &= \lim_{N\to\infty}\sum_{i=1}^{N}\langle 0|e^{\hat{M}(0)(\tau-t_i)}\left\{\left.\frac{\partial \hat{M}(\chi)}{\partial(i\chi)}\right|_{\chi=0}\Delta\tau\right\}e^{\hat{M}(0)\Delta\tau}e^{\hat{M}(0)t_{i-1}}|W(0)\rangle \\ &= \lim_{N\to\infty}\sum_{i=1}^{N}\langle 0|e^{\hat{M}(0)(\tau-t_i)}\left\{\left.\frac{\partial e^{\hat{M}(\chi)\Delta\tau}}{\partial(i\chi)}\right|_{\chi=0}\right\}e^{\hat{M}(0)t_{i-1}}|W(0)\rangle \\ &= \lim_{N\to\infty}\frac{\partial}{\partial(i\chi)}\left.\left\{\sum_{i=1}^{N}\langle 0|e^{\hat{M}(0)(\tau-t_i)}e^{\hat{M}(\chi)\Delta\tau}e^{\hat{M}(0)t_{i-1}}|W(0)\rangle\right\}\right|_{\chi=0} \\ &= \lim_{N\to\infty}\frac{\partial}{\partial(i\chi)}\left.\langle 0|e^{\hat{M}(\chi)\Delta\tau}e^{\hat{M}(\chi)\Delta\tau}\cdots e^{\hat{M}(\chi)\Delta\tau}|W(0)\rangle\right|_{\chi=0} \\ &= \left.\frac{\partial Z_\tau(\chi)}{\partial(i\chi)}\right|_{\chi=0},\end{aligned} \tag{29}$$

where we added a factor $e^{\hat{M}(0)\Delta\tau}$ after $\mathcal{J}_{\rm R}$, which is just a unity since $\Delta\tau e^{\hat{M}(0)\Delta\tau}\sim\Delta\tau+o[(\Delta\tau)^2]$.

Similarly, the second moment is

$$
\begin{aligned}
\langle \hat{n}^2 \rangle_\tau &= \left\langle \int_0^\tau \frac{dt}{-e} I(t) \int_0^\tau \frac{dt'}{-e} I(t') \right\rangle \\
&= \int_0^\tau \frac{dt}{-e} \int_0^t \frac{dt'}{-e} \langle I(t) I(t') \rangle + \int_0^\tau \frac{dt}{-e} \int_t^\tau \frac{dt'}{-e} \langle I(t') I(t) \rangle \\
&= \int_0^\tau \frac{dt}{-e} \int_0^t \frac{dt'}{-e} \langle 0 | e^{\hat{M}(0)(\tau - t)} \mathcal{J}_R e^{\hat{M}(0)(t-t')} \mathcal{J}_R e^{\hat{M}(0)t'} | W(0) \rangle \\
&+ \int_0^\tau \frac{dt}{-e} \int_t^\tau \frac{dt'}{-e} \langle 0 | e^{\hat{M}(0)(\tau - t')} \mathcal{J}_R e^{\hat{M}(0)(t'-t)} \mathcal{J}_R e^{\hat{M}(0)t} | W(0) \rangle \\
&= \lim_{N \to \infty} \sum_{i > j = 1}^{N} \left(\frac{\Delta\tau}{-e} \right)^2 \langle 0 | e^{\hat{M}(0)(\tau - t_i)} \mathcal{J}_R e^{\hat{M}(0)\Delta\tau} e^{\hat{M}(0)(t_{i-1} - t_j)} \mathcal{J}_R e^{\hat{M}(0)\Delta\tau} e^{\hat{M}(0)t_{j-1}} | W(0) \rangle \\
&+ \lim_{N \to \infty} \sum_{j > i = 1}^{N} \left(\frac{\Delta\tau}{-e} \right)^2 \langle 0 | e^{\hat{M}(0)(\tau - t_j)} \mathcal{J}_R e^{\hat{M}(0)\Delta\tau} e^{\hat{M}(0)(t_{j-1} - t_i)} \mathcal{J}_R e^{\hat{M}(0)\Delta\tau} e^{\hat{M}(0)t_{i-1}} | W(0) \rangle \\
&= \lim_{N \to \infty} \sum_{i > j = 1}^{N} \langle 0 | e^{\hat{M}(0)(\tau - t_i)} \left\{ \left. \frac{\partial e^{\hat{M}(\chi)\Delta\tau}}{\partial (i\chi)} \right|_{\chi = 0} \right\} e^{\hat{M}(0)(t_{i-1} - t_j)} \left\{ \left. \frac{\partial e^{\hat{M}(\chi)\Delta\tau}}{\partial (i\chi)} \right|_{\chi = 0} \right\} e^{\hat{M} t_{j-1}} | W(0) \rangle \\
&+ \lim_{N \to \infty} \sum_{j > i = 1}^{N} \langle 0 | e^{\hat{M}(0)(\tau - t_j)} \left\{ \left. \frac{\partial e^{\hat{M}(\chi)\Delta\tau}}{\partial (i\chi)} \right|_{\chi = 0} \right\} e^{\hat{M}(0)(t_{j-1} - t_i)} \left\{ \left. \frac{\partial e^{\hat{M}(\chi)\Delta\tau}}{\partial (i\chi)} \right|_{\chi = 0} \right\} e^{\hat{M} t_{i-1}} | W(0) \rangle \\
&= \lim_{N \to \infty} \frac{\partial^2}{\partial (i\chi)^2} \langle 0 | e^{\hat{M}(\chi)\Delta\tau} e^{\hat{M}(\chi)\Delta\tau} \cdots e^{\hat{M}(\chi)\Delta\tau} | W(0) \rangle \Big|_{\chi = 0} \\
&= \left. \frac{\partial^2 Z_\tau(\chi)}{\partial (i\chi)^2} \right|_{\chi = 0} .
\end{aligned} \tag{30}
$$

In general, for $m \geq 1$,

$$
\left. \frac{\partial^m Z_\tau(\chi)}{\partial (i\chi)^m} \right|_{\chi = 0} = \langle \hat{n}^m \rangle_\tau . \tag{31}
$$

Therefore, in fact, $Z_\tau(\chi)$ is the *electron number generating function*. We define the *cumulant generating function* by

$$
F(\chi) = \lim_{\tau \to \infty} \frac{1}{\tau} \ln Z_\tau(\chi), \tag{32}
$$

and hence we can obtain all order of the cumulant[2] by

$$
\langle \hat{n}^m \rangle_{\tau c} = \tau \left. \frac{\partial^m F(\chi)}{\partial (i\chi)^m} \right|_{\chi = 0} . \tag{33}
$$

Therefore, the term “full-counting statistics (FCS)” had appeared since all the cumulant of n are accessible when we obtain $F(\chi)$.

6.3 Average current and noise power

Here, we formalize the average current and noise power. We assume that the system is in a steady state and then we can freely shift the origin of the time. We introduce a random function $I_\tau(t)$, which is equal

[2]For those who are not familiar with the cumulant, let me show the concrete relations between the moment and cumulant:

$$
\langle \hat{n} \rangle_{\tau c} = \langle \hat{n} \rangle_\tau , \quad \langle \hat{n}^2 \rangle_{\tau c} = \langle \hat{n}^2 \rangle_\tau - \langle \hat{n} \rangle_\tau^2 ,
$$
$$
\langle \hat{n}^3 \rangle_{\tau c} = \langle \hat{n}^3 \rangle_\tau - 3 \langle \hat{n} \rangle_\tau \langle \hat{n}^2 \rangle_\tau + 2 \langle \hat{n} \rangle_\tau^3 , \cdots
$$

to $I(t)$ for $-\tau/2 < t < \tau/2$ and zero otherwise. Hence, its Fourier transform is

$$I_\tau(i\omega) = \int_{-\infty}^{\infty} dt\; I_\tau(t) e^{-i\omega t}, \tag{34}$$

and its zero frequency limit is

$$I_\tau(i\omega = 0) = \int_{-\tau/2}^{\tau/2} dt\; I(t) = -en, \tag{35}$$

where n is the electron number, transferred in a time period τ as discussed in Eq. (19). Then the average current, I, is defined by

$$I = \frac{-e}{\tau} \langle \hat{n} \rangle_\tau = -e \left. \frac{\partial F(\chi)}{\partial (i\chi)} \right|_{\chi=0}. \tag{36}$$

Then we define another random variable $x_\tau(t)$, which is equal to $I(t) - I$ for $-\tau/2 < t < \tau/2$ and zero otherwise. Repeating the arguments in the lecture note on May 20, the Fourier transform is

$$X_\tau(i\omega) = \int_{-\infty}^{\infty} dt x_\tau(t) e^{-i\omega t}, \tag{37}$$

and its (current noise) power spectrum is

$$S_x(\omega) = \lim_{\tau\to\infty} \frac{2}{\tau} \langle |X_\tau(i\omega)|^2 \rangle_\tau . \tag{38}$$

Since, $X_\tau(i\omega = 0) = -e(n - \langle \hat{n} \rangle_\tau)$,

$$\begin{aligned} S_x(0) &= \lim_{\tau\to\infty} \frac{2}{\tau} \langle | -e(\hat{n} - \langle \hat{n} \rangle_\tau)|^2 \rangle_\tau \\ &= \lim_{\tau\to\infty} \frac{2e^2}{\tau} \left\{ \langle \hat{n}^2 \rangle_\tau - \langle \hat{n} \rangle_\tau^2 \right\} \\ &= \lim_{\tau\to\infty} \frac{2e^2}{\tau} \langle \hat{n}^2 \rangle_{\tau c} = 2e^2 \left. \frac{\partial^2 F(\chi)}{\partial (i\chi)^2} \right|_{\chi=0}. \end{aligned} \tag{39}$$

where $\langle \cdots \rangle_{\tau c}$ represents cumulant of some random variable.

Explicit demonstration of the current and noise for the Poisson process (independent electron flow in the vacuum tube considered on May 20) is shown in Appendix A.

6.4 Conclusion

We had introduced the notion of counting statistics of net transferred electron in a period τ with using a counting field χ. Electron number generating function and the cumulant generating function are derived which can provide all moments/cumulants of the net transferred electron number. We will discuss the current and noise of the electrons governed by the master equation (3) in the next lecture on June 3.

A Cumulant generation function of Poisson process

We consider the Poisson distribution characterized by a positive parameter η,

$$P(n) = e^{-\eta} \frac{\eta^n}{n!}. \tag{40}$$

Moment generating function is

$$Z(\chi) = \sum_{n=0}^{\infty} P(n) e^{i\chi n} = e^{-\eta} e^{\eta e^{i\chi}} = e^{\eta(e^{i\chi} - 1)}. \tag{41}$$

We can recover the number distribution function $P(n)$ from $Z(\chi)$ by the transformation

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} d\chi Z(\chi)e^{-i\chi n} &= \frac{1}{2\pi}\int_0^{2\pi} d\chi e^{-i\chi n+\eta(e^{i\chi}-1)} \\ &= \frac{e^{-\eta}}{2\pi}\int_0^{2\pi} d\chi e^{-i\chi n}\sum_{m=0}^{\infty}\frac{(\eta e^{i\chi})^m}{m!} = e^{-\eta}\sum_{m=0}^{\infty}\frac{\eta^m}{m!}\int_0^{2\pi}\frac{d\chi}{2\pi}e^{i\chi(m-n)} \\ &= e^{-\eta}\frac{\eta^n}{n!} = P(n). \end{aligned} \tag{42}$$

Cumulant generation function is

$$F(\chi) = \frac{1}{\tau}\ln Z(\chi) = \frac{\eta}{\tau}(e^{i\chi}-1) = \gamma(e^{i\chi}-1), \tag{43}$$

where in the last equation we set $\eta \equiv \gamma\tau$ with a certain rate γ.

By applying the relations in the main text, the average current is

$$I = -e\left.\frac{\partial F(\chi)}{\partial(i\chi)}\right|_{\chi=0} = -e\gamma, \tag{44}$$

and the current noise is

$$S = 2e^2\left.\frac{\partial^2 F(\chi)}{\partial(i\chi)^2}\right|_{\chi=0} = 2e^2\gamma. \tag{45}$$

Hence, the Fano factor, $\mathcal{F}$, is

$$\mathcal{F} = \frac{S}{2|I|} = e, \tag{46}$$

which is consistent with the analysis at the end of the lecture note on May 20.

References

[1] 齊藤圭司、「非平衡輸送現象：輸送現象における計数統計を学ぶための基礎（講義ノート）」、物性研究 **92**, 345-376 (2009).

[2] 内海裕洋、「量子ドットにおける完全計数統計」固体物理 **41**, 909 (2006).