Basics of Tomonaga-Luttinger liquid Theory

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(Dated: February 14, 2020)

This note reviews the basics of Tomonaga-luttinger liquid based on the review paper by Jan von Delft and Herbert Scholler, "Bosonization for beginners - refermionization for experts", Ann. Phys. 7, 225-306 (1998) and my presentation slides for the ERATO Tarucha Project, "Tomonaga-Luttinger liquid - Bosonization of one-dimensional electron and transport -", the first part: "Constructive approach of Bosonization of one-dimensional electron system", on May 17, 2001. The purpose for rewriting this note is to clarify the notation and detailed derivations of (chiral) Tomonaga-Luttinger liquid, which is now recently been activated again in the context of "quantum quench".

PACS numbers:

I. SYSTEM

We will discuss an ensemble of Fermionic particles with a mode η confined in a one-dimensional system of length L. The mode η can be right/left going chiral modes, spin up/down modes, the ground, the first excited, and higher excited one-dimensional subbands, and various edge channels formed in a two-dimensional system under a large parpenducular magnetic field. For simplicity, first we will discuss only one of the modes η and disregard the interaction within the mode η for the time being. The effect of the interaction is discussed in Sec. VI B.

Let us define the field operator, $\hat{\Psi}_\eta(x)$, which annihilates one Fermion particle of mode η at $x \in [-\frac{L}{2}, \frac{L}{2})$, where L is the length of the system. This is constructed by the Fermionic annihilation operator, $\hat{c}_k$, as

$$\hat{\Psi}_\eta(x) \equiv \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} e^{-ikx} \hat{c}_k, \tag{1}$$

where the wave number index k is determined by the boundary condition of the system. We undertake following boundary condition

$$\hat{\Psi}_\eta\left(x + \frac{L}{2}\right) = e^{i\pi\delta_b} \hat{\Psi}_\eta\left(x - \frac{L}{2}\right), \tag{2}$$

where a real constant δ_b is in the range $[0, 2)$. Then clearly, the wave numbers are discrete as

$$k = \frac{2\pi}{L}\left(n_k - \frac{\delta_b}{2}\right), \tag{3}$$

where n_k is an integer and hence $k \in [-\infty, \infty]$. The field operator, Eq.(1), containing the factor $\exp[-ikx]$, represents a left propagating wave. We assume the energy of the state k as ϵ_k. It is important to note that in the *Luttinger model*, the energy ϵ_k is a monotonic function of k, while the conventional metallic system shows parabolic dependence $\epsilon_k \propto k^2$. The fact that there is no lowest energy state is essential for the following discussion, although this may sound unphysical. However, most of physical properties in the metallic system at low temperatures (typically $T \ll T_F$, where T_F is the Fermi temperature determined by the electron density, effective mass, and so forth.) are only concerned with the states near the Fermi points and this assumption (on states with infinitely negative energies) does not introduce problems. We will discuss the relation between this Luttinger model and real physical system in Sec.VI.

The Fermionic creation operator of state k, $\hat{c}_k^\dagger$, also defines another field operator,

$$\hat{\Psi}_\eta^\dagger(x) \equiv \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} e^{ikx} \hat{c}_k^\dagger. \tag{4}$$

With the fermionic anti-commutation relation

$$\{\hat{c}_k, \hat{c}_{k'}^\dagger\} \equiv \hat{c}_k \hat{c}_{k'}^\dagger + \hat{c}_{k'}^\dagger \hat{c}_k = \delta_{kk'}, \tag{5}$$

we have the anti-commutation relation of the field operators,

$$\begin{aligned}
\{\hat{\Psi}_\eta(x), \hat{\Psi}_\eta^\dagger(x')\} &= \frac{1}{L}\sum_{k,k'} e^{-ikx+ik'x'} \{\hat{c}_k, \hat{c}_{k'}\} \\
&= \frac{1}{L}\sum_k e^{ik(x'-x)} \\
&\equiv \frac{1}{L}\sum_{n_k=-\infty}^{\infty} \exp\left[\frac{2\pi i}{L}\left(n_k - \frac{\delta_b}{2}\right)(x - x')\right] \\
&= \frac{1}{L} e^{-\frac{\pi\delta_b i}{L}(x-x')} \sum_{n_k=-\infty}^{\infty} \exp\left[\frac{2\pi i}{L}(x - x')n_k\right] \\
&= \frac{2\pi}{L} e^{-\frac{\pi\delta_b i}{L}(x-x')} \sum_{m=-\infty}^{\infty} \delta\left(\frac{2\pi(x - x')}{L} - 2\pi m\right) \\
&= \frac{2\pi}{L} e^{-\frac{\pi\delta_b i}{L}(x-x')} \frac{L}{2\pi} \sum_{m=-\infty}^{\infty} \delta\left(x - x' - Lm\right) \\
&= \delta(x - x').
\end{aligned} \tag{6}$$

We used the identity

$$\sum_{n=-\infty}^{\infty} e^{iny} = 2\pi \sum_{m=-\infty}^{\infty} \delta(y - 2\pi m), \tag{7}$$

where n, m are integers and the property of the delta-function $\delta(ax) = \frac{1}{a}\delta(x)$. In the last equation, we chose $m = 0$ since $x, x' \in [-\frac{L}{2}, \frac{L}{2})$. All other operators are anti-commute, like $\{\hat{c}_k, \hat{c}_{k'}\} = \{\hat{c}_k^\dagger, \hat{c}_{k'}^\dagger\} = 0$ and hence $\{\hat{\Psi}_\eta(x), \hat{\Psi}_\eta(x')\} = \{\hat{\Psi}_\eta^\dagger(x), \hat{\Psi}_\eta^\dagger(x')\} = 0$. Please also note that we can construct the annihilation operator from the field operator,

$$\hat{c}_k \;=\; \frac{1}{\sqrt{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\; e^{ikx} \hat{\Psi}_\eta(x), \tag{8}$$

and similar expression for $\hat{c}_k^\dagger$ by $\hat{\Psi}_\eta^\dagger(x)$.

II. FOCK SPACE $\mathcal{F}$

Let us consider a Hilbert space $\mathcal{H}_N$ with a fixed particle number N. The Fock space is the sum for all N,

$$\mathcal{F} \;\equiv\; \sum_{\oplus N} \mathcal{H}_N. \tag{9}$$

We define the vacuum state $|0\rangle_0$, which satisfies

$$\hat{c}_k |0\rangle_0 = 0, \;\text{ for } k > 0, \tag{10}$$
$$\hat{c}_k^\dagger |0\rangle_0 = 0, \;\text{ for } k \le 0, \tag{11}$$

where we chose $k = 0$ arbitrary, which is allowed because the wave number is unbounded. In the following argument, we fix our choice of the origin of k. Physically, this vacuum state corresponds to the situation that all the states with $n_k = 0, -1, -2, \ldots$ are occupied by the particles and all the states with $n_k = 1, 2, 3, \ldots$ are empty. Let us define normal order operation $\overset{*}{*} \cdots \overset{*}{*}$ relative to $|0\rangle_0$ by

$$\overset{*}{*}\, \hat{A}\hat{B}\hat{C} \cdots \overset{*}{*} \;\equiv\; \hat{A}\hat{B}\hat{C} \cdots - {}_0\langle 0|\, \hat{A}\hat{B}\hat{C} \cdots |0\rangle_0\,, \tag{12}$$

where $\hat{A}, \hat{B}, \hat{C}, \ldots$ are operators like $\hat{c}_k$ or $\hat{c}_{k'}^\dagger$. This operation is essential to avoid the infinity by subtracting the expectation value with respect to the vacuum state $|0\rangle_0$. The number operator is defined by

$$\hat{N} \;\equiv\; \sum_{k=-\infty}^{\infty} \overset{*}{*}\, \hat{c}_k^\dagger \hat{c}_k \,\overset{*}{*}\,. \tag{13}$$

Clearly the eigenvalue (number) of the vacuum state $|0\rangle_0$ is zero,

$$\begin{aligned} \hat{N} |0\rangle_0 \;&=\; \sum_{k=-\infty}^{\infty} \left(\hat{c}_k^\dagger \hat{c}_k - {}_0\langle 0|\, \hat{c}_k^\dagger \hat{c}_k |0\rangle_0 \right) |0\rangle_0 \\ &=\; \sum_k \hat{c}_k^\dagger \hat{c}_k |0\rangle_0 - \sum_{k=-\infty}^{0} |0\rangle_0 \\ &=\; 0. \end{aligned} \tag{14}$$

Let us consider the set of all states, which have the eigenvalue N of the operator $\hat{N}$. This set spans the Hilbert space $\mathcal{H}_N$. Then we define N-particle vacuum state $|N\rangle_0$. More explicitly, by defining $k_{0N} \equiv \frac{2\pi}{L}(N - \frac{\delta_b}{2})$,

$$\hat{c}_k |N\rangle_0 = 0, \;\text{ for } k > k_{0N}, \tag{15}$$
$$\hat{c}_k^\dagger |N\rangle_0 = 0, \;\text{ for } k \le k_{0N}. \tag{16}$$

It can be proved that all states in $\mathcal{H}_N$ can be generated from $|N\rangle_0$ by particle-hole pair excitations,

$$|N\rangle_{\bar{f}} \;=\; \bar{f}(\{\hat{c}_k^\dagger \hat{c}_{k'}\})\, |N\rangle_0\,, \tag{17}$$

where $\bar{f}(\{\hat{c}_k^\dagger \hat{c}_{k'}\})$ is the function of the products $\hat{c}_k^\dagger \hat{c}_{k'}$ with arbitrary set k, k'. Examples of these for $N = 2$ are $\hat{c}_3^\dagger \hat{c}_1 |2\rangle_0$ and $\hat{c}_4^\dagger \hat{c}_{-1} \hat{c}_3^\dagger \hat{c}_1 |2\rangle_0$.

III. BOSONIC REPRESENTATION OF FOCK SPACE

For $q = 2\pi n_q / L > 0$, we define following Bosonic creation and annihilation operators from a pair of Fermionic creating and annihilation operators,

$$\begin{aligned} \hat{b}_q^\dagger \;&=\; \frac{i}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} \hat{c}_{k+q}^\dagger \hat{c}_k, \\ \hat{b}_q \;&=\; \frac{-i}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} \hat{c}_{k-q}^\dagger \hat{c}_k, \end{aligned} \tag{18}$$

where the normalization factor $\pm i/\sqrt{n_q}$ is chosen to simplify the following arguments. Note $q = 0$ ($n_q = 0$) is excluded from the discussions. $\hat{b}_q^\dagger |N\rangle_{\bar{f}}$ ($\hat{b}_q |N\rangle_{\bar{f}}$) is a linear combination of the states with momentum q particle-hole pair created (annihilated) from $|N\rangle_{\bar{f}}$. Clearly, $\hat{b}_q |N\rangle_0 = 0$ for $\forall q > 0$ since there is no particle-hole excited states to annihilate.

Let us check various commutation relations of $\hat{b}_q$ and $\hat{b}_q^\dagger$:

$$\begin{aligned} [\hat{N}, \hat{b}_q^\dagger] \;&=\; \left[\sum_k \overset{*}{*}\, \hat{c}_k^\dagger \hat{c}_k \,\overset{*}{*}, \frac{i}{\sqrt{n_q}} \sum_{k'} \hat{c}_{k'+q}^\dagger \hat{c}_{k'} \right] \\ &=\; \frac{i}{\sqrt{n_q}} \sum_{k,k'} [\hat{c}_k^\dagger \hat{c}_k, \hat{c}_{k'+q}^\dagger \hat{c}_{k'}], \end{aligned} \tag{19}$$

where the commutator reduces to

$$\begin{aligned} [\hat{c}_k^\dagger \hat{c}_k, \hat{c}_{k'+q}^\dagger \hat{c}_{k'}] \;&=\; \hat{c}_k^\dagger [\hat{c}_k, \hat{c}_{k'+q}^\dagger \hat{c}_{k'}] + [\hat{c}_k^\dagger, \hat{c}_{k'+q}^\dagger \hat{c}_{k'}] \hat{c}_k \\ &=\; \hat{c}_k^\dagger \left(\{\hat{c}_k, \hat{c}_{k'+q}^\dagger\} \hat{c}_{k'} - \hat{c}_{k'+q}^\dagger \{\hat{c}_k, \hat{c}_{k'}\} \right) \\ &\quad + \left(\{\hat{c}_k^\dagger, \hat{c}_{k'+q}^\dagger\} \hat{c}_{k'} - \hat{c}_{k'+q}^\dagger \{\hat{c}_k^\dagger, \hat{c}_{k'}\} \right) \hat{c}_k \\ &=\; \delta_{k,k'+q} \hat{c}_k^\dagger \hat{c}_{k'} - \delta_{k,k'} \hat{c}_{k'+q}^\dagger \hat{c}_k. \end{aligned}$$

We have used the relation $[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$ and $[\hat{A},\hat{B}\hat{C}] = \{\hat{A},\hat{B}\}\hat{C} - \hat{B}\{\hat{A},\hat{C}\}$ with arbitrary (Bosonic or Fermionic) operators $\hat{A},\hat{B},\hat{C}$. Therefore, the commutator is

$$\begin{aligned}[\hat{N},\hat{b}_q^\dagger] &= \frac{i}{\sqrt{n_q}}\sum_k\left(\hat{c}_k^\dagger\hat{c}_{k-q} - \hat{c}_{k+q}^\dagger\hat{c}_k\right)\\ &= \frac{i}{\sqrt{n_q}}\sum_k \overset{*}{*}\left(\hat{c}_k^\dagger\hat{c}_{k-q} - \hat{c}_{k+q}^\dagger\hat{c}_k\right)\overset{*}{*} = 0, \end{aligned} \quad (20)$$

where taking the normal order $\overset{*}{*}\cdots\overset{*}{*}$ is allowed since $q>0$ and we changed the variable $k\to k+q$ in the first term which is possible in the normal order. Similarly,

$$[\hat{N},\hat{b}_q] = 0. \quad (21)$$

Next,

$$\begin{aligned}[\hat{b}_q,\hat{b}_{q'}] &= \frac{-1}{\sqrt{n_q n_{q'}}}\sum_{k,k'}[\hat{c}_{k-q}^\dagger\hat{c}_k,\hat{c}_{k'-q'}^\dagger\hat{c}_{k'}]\\ &= \frac{-1}{\sqrt{n_q n_{q'}}}\sum_{k,k'}\left(\delta_{k,k'-q'}\hat{c}_{k-q}^\dagger\hat{c}_{k'} - \delta_{k-q,k'}\hat{c}_{k'-q'}^\dagger\hat{c}_k\right)\\ &= \frac{-1}{\sqrt{n_q n_{q'}}}\sum_k\left(\hat{c}_{k-q}^\dagger\hat{c}_{k+q'} - \hat{c}_{k-q-q'}^\dagger\hat{c}_k\right)\\ &= \frac{-1}{\sqrt{n_q n_{q'}}}\sum_k\overset{*}{*}\left(\hat{c}_{k-q}^\dagger\hat{c}_{k+q'} - \hat{c}_{k-q-q'}^\dagger\hat{c}_k\right)\overset{*}{*}\\ &= 0, \end{aligned} \quad (22)$$

where we took the normal order since $q,q'>0$ and in the last equation we changed the variable $k\to k+q'$ in the second term. Similarly, we have

$$[\hat{b}_q^\dagger,\hat{b}_{q'}^\dagger] = 0. \quad (23)$$

Finally,

$$\begin{aligned}[\hat{b}_q,\hat{b}_{q'}^\dagger] &= \frac{1}{\sqrt{n_q n_{q'}}}\sum_{k,k'}[\hat{c}_{k-q}^\dagger\hat{c}_k,\hat{c}_{k'+q'}^\dagger\hat{c}_{k'}]\\ &= \frac{1}{\sqrt{n_q n_{q'}}}\sum_{k,k'}\left(\delta_{k,k'+q'}\hat{c}_{k-q}^\dagger\hat{c}_{k'} - \delta_{k-q,k'}\hat{c}_{k'+q'}^\dagger\hat{c}_k\right)\\ &= \frac{1}{\sqrt{n_q n_{q'}}}\sum_k\left(\hat{c}_{k-q}^\dagger\hat{c}_{k-q'} - \hat{c}_{k-q+q'}^\dagger\hat{c}_k\right). \end{aligned}$$

Here we should be careful for the conditions of q and q'. If $q\neq q'$, vacuum expectation values of these terms are zero and

$$\begin{aligned}[\hat{b}_q,\hat{b}_{q'}^\dagger]_{q\neq q'} &= \frac{1}{\sqrt{n_q n_{q'}}}\sum_k\overset{*}{*}\left(\hat{c}_{k-q}^\dagger\hat{c}_{k-q'} - \hat{c}_{k-q+q'}^\dagger\hat{c}_k\right)\overset{*}{*}\\ &= 0, \end{aligned} \quad (24)$$

where we changed variable $k\to k-q'$ in the second term.

In contrast, for $q=q'$,

$$\begin{aligned}[\hat{b}_q,\hat{b}_{q'}^\dagger]_{q=q'} &= \frac{1}{n_q}\sum_k\left(\hat{c}_{k-q}^\dagger\hat{c}_{k-q} - \hat{c}_k^\dagger\hat{c}_k\right)\\ &= \frac{1}{n_q}\sum_k\Big(\overset{*}{*}\hat{c}_{k-q}^\dagger\hat{c}_{k-q}\overset{*}{*} + {}_0\langle 0|\hat{c}_{k-q}^\dagger\hat{c}_{k-q}|0\rangle_0\\ &\quad - \overset{*}{*}\hat{c}_k^\dagger\hat{c}_k\overset{*}{*} - {}_0\langle 0|\hat{c}_k^\dagger\hat{c}_k|0\rangle_0\Big)\\ &= \frac{1}{n_q}\left(\hat{N} + \sum_{k=-\infty}^{q} - \hat{N} - \sum_{k=-\infty}^{0}\right)\\ &= \frac{1}{n_q}\times n_q = 1. \end{aligned} \quad (25)$$

Therefore, we have the canonical commutation relation of Bosons,

$$[\hat{b}_q,\hat{b}_{q'}^\dagger] = \delta_{qq'}. \quad (26)$$

For all N, $|N\rangle_0$ is the Boson vacuum state, since $\hat{b}_q|N\rangle_0 = 0$, $\forall q$. In contrast, $\hat{b}_q^\dagger|N\rangle_0$ represents collectively excited state with momentum q. In particular situation of linear energy dispersion, all the excitations are energetically degenerate. It can be proved that all the states generated by $\hat{b}_q^\dagger$ from $|N\rangle_0$, like $|N\rangle_f \equiv f(\{\hat{b}_q^\dagger\})|N\rangle_0$ span the complete N-particle Hilbert space $\mathcal{H}_N$ generated by Fermion operators, $\bar{f}(\{\hat{c}_k^\dagger\hat{c}_{k'}\})|N\rangle_0$. Here, we skip the proof, please refer Ref.[1] if one is interested.

Since the operators $\hat{b}_q,\hat{b}_q^\dagger$ conserves the particle number N, we introduce Klein factors $\hat{F},\hat{F}^\dagger$ to connect the states with different N and to give a correct anti-commutation relations with different modes. We define

$$\begin{aligned}\hat{F}^\dagger|N\rangle_f &= \hat{F}^\dagger f(\{\hat{b}_q^\dagger\})|N\rangle_0\\ &\equiv f(\{\hat{b}_q^\dagger\})\hat{c}_{k_{0N+1}}^\dagger|N\rangle_0\\ &= f(\{\hat{b}_q^\dagger\})|N+1\rangle_0. \end{aligned} \quad (27)$$

Similarly,

$$\hat{F}|N\rangle_f \equiv f(\{\hat{b}_q^\dagger\})|N-1\rangle_0. \quad (28)$$

Clearly,

$$\begin{aligned}\hat{F}^\dagger\hat{F}|N\rangle_f &= \hat{F}^\dagger f(\{\hat{b}_q^\dagger\})|N-1\rangle_0 = f(\{\hat{b}_q^\dagger\})|N\rangle_0 = |N\rangle_f,\\ \hat{F}\hat{F}^\dagger|N\rangle_f &= |N\rangle_f. \end{aligned} \quad (29)$$

We have required commutation relations $[\hat{b}_q,\hat{F}^\dagger] = [\hat{b}_q^\dagger,\hat{F}^\dagger] = [\hat{b}_q,\hat{F}] = [\hat{b}_q^\dagger,\hat{F}] = 0$. If we extend the discussions with different modes $\eta\neq\eta'$, $\{\hat{F}_\eta^\dagger,\hat{F}_{\eta'}\} = \{\hat{F}_\eta,\hat{F}_{\eta'}\} = \{\hat{F}_\eta^\dagger,\hat{F}_{\eta'}^\dagger\} = 0$, which will correctly treat the anti-commutation relations of the field operators introduced in the following sections.

IV. BOSON FIELD AND BOSONIZATION OF FERMION FIELD

A. Boson field operators

Let us introduce Boson field operators as follows

$$\hat{\varphi}_\eta(x) \equiv -\sum_{q>0} \frac{1}{\sqrt{n_q}} e^{-iqx} \hat{b}_q e^{-\frac{aq}{2}}, \tag{30}$$

$$\hat{\varphi}^\dagger_\eta(x) \equiv -\sum_{q>0} \frac{1}{\sqrt{n_q}} e^{iqx} \hat{b}^\dagger_q e^{-\frac{aq}{2}}, \tag{31}$$

where a is the infinitesimal positive length cut-off parameter. Real (Hermite) field is defined with these boson field operators,

$$\hat{\Phi}_\eta(x) = \hat{\varphi}_\eta(x) + \hat{\varphi}^\dagger_\eta(x), \tag{32}$$

which has a clear physical meaning by following discussions. Particle density operator at x is defined by

$$\begin{aligned}\hat{\rho}(x) &= {}^*_* \hat{\Psi}^\dagger_\eta(x)\hat{\Psi}_\eta(x) {}^*_* \\ &= \frac{1}{L}\sum_{k,k'} e^{ik'x-ikx} {}^*_* \hat{c}^\dagger_{k'}\hat{c}_k {}^*_* .\end{aligned} \tag{33}$$

Be changing variable $k' \to k-q$, we have

$$\begin{aligned}\hat{\rho}(x) &= \frac{1}{L}\sum_{q=-\infty}^{\infty} e^{-iqx} \sum_{k=-\infty}^{\infty} {}^*_* \hat{c}^\dagger_{k-q}\hat{c}_k {}^*_* \\ &= \frac{1}{L}\sum_{q>0} i\sqrt{n_q} e^{-iqx} \frac{-i}{\sqrt{n_q}} \sum_k \hat{c}^\dagger_{k-q}\hat{c}_k \\ &\quad + \frac{1}{L}\sum_k {}^*_* \hat{c}^\dagger_k \hat{c}_k {}^*_* \\ &\quad + \frac{1}{L}\sum_{q<0} (-i\sqrt{|n_q|}) e^{-iqx} \frac{i}{\sqrt{|n_q|}} \sum_k \hat{c}^\dagger_{k-q}\hat{c}_k \\ &= \frac{1}{L}\sum_{q>0} i\sqrt{n_q}(e^{-iqx}\hat{b}_q - e^{iqx}\hat{b}^\dagger_q) + \frac{\hat{N}}{L}.\end{aligned} \tag{34}$$

While if we take the derivative of $\hat{\Phi}_\eta(x)$,

$$\begin{aligned}\partial_x\hat{\Phi}_\eta(x) &= -\sum_{q>0}\frac{1}{\sqrt{n_q}}(-iqe^{-iqx}\hat{b}_q + iqe^{iqx}\hat{b}^\dagger_q)e^{-\frac{aq}{2}} \\ &= -\sum_{q>0}\frac{-iq}{\sqrt{n_q}}(e^{-iqx}\hat{b}_q - e^{iqx}\hat{b}^\dagger_q)e^{-\frac{aq}{2}} \\ &= \frac{2\pi}{L}\sum_{q>0} i\sqrt{n_q}(e^{-iqx}\hat{b}_q - e^{iqx}\hat{b}^\dagger_q)e^{-\frac{aq}{2}}.\end{aligned} \tag{35}$$

Therefore, in the limit of $a \to 0^+$,

$$\frac{1}{2\pi}\partial_x\hat{\Phi}_\eta(x) = \hat{\rho}(x) - \frac{\hat{N}}{L}, \tag{36}$$

and since $\hat{N}/L$ represents average particle line density, $\partial_x\hat{\Phi}_\eta(x)/(2\pi)$ shows the fluctuation of the particle density. In particular, kinks of the amplitude of 2π in the function $\hat{\Phi}_\eta(x)$ represents localized ± 1 charge fluctuations.

Let us examine the commutation relations of Boson field operators, clearly

$$[\hat{\varphi}_\eta(x), \hat{\varphi}_\eta(x')] = [\hat{\varphi}^\dagger_\eta(x), \hat{\varphi}^\dagger_\eta(x)] = 0, \tag{37}$$

and

$$\begin{aligned}[\hat{\varphi}_\eta(x), \hat{\varphi}^\dagger_\eta(x')] &= \sum_{q,q'>0}\frac{1}{\sqrt{n_q n'_q}} e^{i(q'x'-qx)} e^{-\frac{a}{2}(q+q')}[\hat{b}_q, \hat{b}^\dagger_{q'}] \\ &= \sum_{q>0}\frac{1}{n_q} e^{iq(x'-x)-aq} \\ &= \sum_{n>0}\frac{1}{n} e^{-i\frac{2\pi}{L}(x-x'-ia)n} \\ &= -\ln\left[1 - e^{-i\frac{2\pi}{L}(x-x'-ia)}\right],\end{aligned} \tag{38}$$

where we used the series expansion $\ln(1-a) = -\sum_{n>0}\frac{a^n}{n}$. Using this relation, we evaluate spatial derivative of the commutation relation $[\hat{\Phi}_\eta(x), \hat{\Phi}_\eta(x')]$,

$$\begin{aligned}&\partial_{x'}[\hat{\Phi}_\eta(x), \hat{\Phi}_\eta(x')] \\ &= \partial_{x'}\{[\hat{\varphi}_\eta(x), \hat{\varphi}^\dagger_\eta(x')] + [\hat{\varphi}^\dagger_\eta(x), \hat{\varphi}_\eta(x')]\} \\ &= \partial_{x'}\{-\ln\left[1 - e^{-i\frac{2\pi}{L}(x-x'-ia)}\right] + \ln\left[1 - e^{-i\frac{2\pi}{L}(x'-x-ia)}\right]\} \\ &= i\frac{2\pi}{L}\left[\frac{e^{-i\frac{2\pi}{L}(x-x'-ia)}}{1 - e^{-i\frac{2\pi}{L}(x-x'-ia)}} + \frac{e^{-i\frac{2\pi}{L}(x'-x-ia)}}{1 - e^{-i\frac{2\pi}{L}(x'-x-ia)}}\right] \\ &= i\frac{2\pi}{L}\left[\frac{1}{e^{i\frac{2\pi}{L}(x-x'-ia)} - 1} + \frac{1}{e^{i\frac{2\pi}{L}(x'-x-ia)} - 1}\right].\end{aligned} \tag{39}$$

Now we take the limit $L \to \infty$ with keeping $x - x'$ finite. Using the relation for $|\epsilon| \ll 1$,

$$\begin{aligned}\frac{1}{e^{i\epsilon}-1} &= \frac{1}{(1 + i\epsilon + \frac{1}{2!}(i\epsilon)^2 + \cdots) - 1} \\ &= \frac{1}{i\epsilon(1 + \frac{i\epsilon}{2} + \cdots)} \sim \frac{1}{i\epsilon}(1 - \frac{i\epsilon}{2}\cdots) \\ &\sim \frac{1}{i\epsilon} - \frac{1}{2} + \mathcal{O}(\epsilon),\end{aligned} \tag{40}$$

we have the limiting form,

$$\begin{aligned}&\partial_{x'}[\hat{\Phi}_\eta(x), \hat{\Phi}_\eta(x')] \\ &\sim i\frac{2\pi}{L}\left[\frac{1}{i\frac{2\pi}{L}(x-x'-ia)} + \frac{1}{i\frac{2\pi}{L}(x'-x-ia)} - 1\right] \\ &= \frac{2ia}{(x-x')^2 + a^2} - i\frac{2\pi}{L} \\ &= 2\pi i\left[\frac{a/\pi}{(x-x')^2 + a^2} - \frac{1}{L}\right].\end{aligned} \tag{41}$$

Finally, taking the limit $a \to 0^+$, we obtain

$$\partial_{x'}[\hat{\Phi}_\eta(x), \hat{\Phi}_\eta(x')] \sim 2\pi i \left[\delta(x-x') - \frac{1}{L}\right]. \quad (42)$$

In the literatures, following conventional form is often used,

$$[\hat{\Phi}_\eta(x), \hat{\Phi}_\eta(x')] = -\pi i \mathrm{Sgn}(x-x'), \quad (43)$$

where Sgn(s) function is 1 for $s > 0$, -1 for $s < 0$ and 0 for $s = 0$.

B. Fermion field operator by boson field

In this subsection, we express Fermion field operator $\hat{\Psi}_\eta(x)$ with boson field operators $\hat{\varphi}_\eta(x), \hat{\varphi}_\eta^\dagger(x)$, and $\hat{\Phi}_\eta(x)$. First we check the commutation relation

$$\begin{aligned}
[\hat{b}_q, \hat{c}_k] &= \frac{-i}{\sqrt{n_q}} \sum_{k'} [\hat{c}^\dagger_{k'-q}\hat{c}_{k'}, \hat{c}_k] \\
&= \frac{-i}{\sqrt{n_q}} \sum_{k'} \left(\hat{c}^\dagger_{k'-q}\{\hat{c}_{k'}, \hat{c}_k\} - \{\hat{c}^\dagger_{k'-q}, \hat{c}_k\}\hat{c}_{k'}\right) \\
&= \frac{i}{\sqrt{n_q}} \sum_{k'} \delta_{k'-q,k}\hat{c}_{k'} \\
&= \frac{i}{\sqrt{n_q}} \hat{c}_{k+q}. \quad (44)
\end{aligned}$$

Therefore,

$$\begin{aligned}
[\hat{b}_q, \hat{\Psi}_\eta(x)] &= \frac{1}{\sqrt{L}} \sum_k e^{-ikx}[\hat{b}_q, \hat{c}_k] \\
&= \frac{1}{\sqrt{L}} \sum_k e^{-ikx} \frac{i}{\sqrt{n_q}} \hat{c}_{k+q} \\
&= \frac{i}{\sqrt{n_q}} \frac{1}{\sqrt{L}} \sum_k e^{-i(k-q)x} \hat{c}_k \\
&= \frac{i}{\sqrt{n_q}} e^{iqx} \frac{1}{\sqrt{L}} \sum_k e^{-ikx} \hat{c}_k \\
&= \alpha_q(x)\hat{\Psi}_\eta(x), \quad (45)
\end{aligned}$$

where we defined a complex function

$$\alpha_q(x) = \frac{i}{\sqrt{n_q}} e^{iqx}, \quad (46)$$

and used the relation $[\hat{A}\hat{B}, \hat{C}] = \hat{A}\{\hat{B}, \hat{C}\} - \{\hat{A}, \hat{C}\}\hat{B}$. Similarly,

$$\begin{aligned}
[\hat{b}_q^\dagger, \hat{c}_k] &= \frac{i}{\sqrt{n_q}} \sum_{k'} [\hat{c}^\dagger_{k'+q}\hat{c}_{k'}, \hat{c}_k] \\
&= \frac{i}{\sqrt{n_q}} \sum_{k'} \left(\hat{c}^\dagger_{k'+q}\{\hat{c}_{k'}, \hat{c}_k\} - \{\hat{c}^\dagger_{k'+q}, \hat{c}_k\}\hat{c}_{k'}\right) \\
&= \frac{-i}{\sqrt{n_q}} \sum_{k'} \delta_{k'+q,k}\hat{c}_{k'} \\
&= \frac{-i}{\sqrt{n_q}} \hat{c}_{k-q}. \quad (47)
\end{aligned}$$

Therefore,

$$\begin{aligned}
[\hat{b}_q^\dagger, \hat{\Psi}_\eta(x)] &= \frac{1}{\sqrt{L}} \sum_k e^{-ikx}[\hat{b}_q^\dagger, \hat{c}_k] \\
&= \frac{1}{\sqrt{L}} \sum_k e^{-ikx} \frac{-i}{\sqrt{n_q}} \hat{c}_{k-q} \\
&= \frac{-i}{\sqrt{n_q}} \frac{1}{\sqrt{L}} \sum_k e^{-i(k+q)x} \hat{c}_k \\
&= \frac{-i}{\sqrt{n_q}} e^{-iqx} \frac{1}{\sqrt{L}} \sum_k e^{-ikx} \hat{c}_k \\
&= \alpha_q^*(x)\hat{\Psi}_\eta(x). \quad (48)
\end{aligned}$$

Let us apply the operator Eq.(45) to the vacuum of N particles $|N\rangle_0$,

$$[\hat{b}_q, \hat{\Psi}_\eta(x)] |N\rangle_0 = \alpha_q(x)\hat{\Psi}_\eta(x) |N\rangle_0, \quad (49)$$

where the left-hand side is

$$\hat{b}_q\hat{\Psi}_\eta(x) |N\rangle_0 - \hat{\Psi}_\eta(x)\hat{b}_q |N\rangle_0 = \hat{b}_q\hat{\Psi}_\eta(x) |N\rangle_0. \quad (50)$$

Therefore, the state $|\alpha_N(x)\rangle \equiv \hat{\Psi}_\eta(x) |N\rangle_0$ is the eigenfunction of the bosonic annihilation operator b_q with the eigenvalue $\alpha_q(x)$,

$$\hat{b}_q |\alpha_N(x)\rangle = \alpha_q(x) |\alpha_N(x)\rangle. \quad (51)$$

It is well-known that such a state is a *coherent state* expressed with $\hat{b}_q^\dagger$,

$$|\alpha_N(x)\rangle = \lambda e^{\sum_{q>0} \alpha_q(x)\hat{b}_q^\dagger} |N\rangle_0, \quad (52)$$

where λ is a phase factor which commutes with $\hat{b}_q$ and $\hat{b}_q^\dagger$. This can be proved as follows: by setting $\hat{A} \equiv \hat{b}_q$ and $\hat{B} \equiv \sum_{q>0} \alpha_q(x)\hat{b}_q^\dagger$, the commutator

$$[\hat{A}, \hat{B}] = \sum_{q'>0} \alpha_{q'}(x)[\hat{b}_q, \hat{b}_{q'}^\dagger] = \alpha_q(x), \quad (53)$$

is a c-number and by applying Eq.(A1), we have

$$[\hat{b}_q, e^{\sum_{q>0} \alpha_q(x)\hat{b}_q^\dagger}] = \alpha_q(x)e^{\sum_{q>0} \alpha_q(x)\hat{b}_q^\dagger}. \quad (54)$$

We apply this operator to the state $\lambda |N\rangle_0$, then we obtain

$$\hat{b}_q |\alpha_N(x)\rangle = \alpha_q(x) |\alpha_N(x)\rangle. \quad (55)$$

(Q.E.D.)

Introducing Klein factor, we set the state $|\alpha_N(x)\rangle$ as

$$\begin{aligned}
\hat{\Psi}_\eta(x) |N\rangle_0 &= \hat{F}\lambda(x)e^{\sum_{q>0} \alpha_q(x)\hat{b}_q^\dagger} |N\rangle_0 \\
&= \hat{F}\lambda(x)e^{-i\hat{\varphi}_\eta^\dagger(x)} |N\rangle_0. \quad (56)
\end{aligned}$$

The factor $\lambda(x)$ is determined as follows: from Fermionic representation,

$$
\begin{aligned}
{}_0\langle N|\,\hat{F}^\dagger\hat{\Psi}_\eta(x)\,|N\rangle_0 &= {}_0\langle N|\,\hat{F}^\dagger\frac{1}{\sqrt{L}}\sum_{k_{0N}\le 0}e^{-ikx}c_k\,|N\rangle_0 \\
&= \frac{1}{\sqrt{L}}e^{-ik_{0N}x}, \qquad (57)
\end{aligned}
$$

where $k_{0N}=\frac{2\pi}{L}(N-\frac{\delta_b}{2})$. Using Bosonic representation Eq.(56), we get

$$
\begin{aligned}
{}_0\langle N|\,\hat{F}^\dagger\hat{\Psi}_\eta(x)\,|N\rangle_0 &= {}_0\langle N|\,\hat{F}^\dagger\hat{F}\lambda(x)e^{-i\hat{\varphi}_\eta^\dagger(x)}\,|N\rangle_0 \\
&= {}_0\langle N|\,e^{-i\hat{\varphi}_\eta^\dagger(x)}\lambda(x)\,|N\rangle_0 \\
&= \lambda(x), \qquad (58)
\end{aligned}
$$

where we used the relation ${}_0\langle N|\,e^{-i\hat{\varphi}_\eta^\dagger(x)}={}_0\langle N|$ since ${}_0\langle N|\,\hat{\varphi}_\eta^\dagger(x)=0$. Hence,

$$
\lambda(x) = \frac{1}{\sqrt{L}}e^{-ik_{0N}x}. \qquad (59)
$$

Now, we had the bosonic expression of $\hat{\Psi}_\eta(x)$, which gives the same state if it is applied to $|N\rangle_0$.

Then we search more general bosonic expression of $\hat{\Psi}_\eta(x)$, which gives the same state if it is applied to a general N particle state $|N\rangle_f\equiv f(\{\hat{b}_q^\dagger\})\,|N\rangle_0$. By setting $\hat{A}=\hat{b}_q^\dagger$ and $\hat{B}=\hat{\Psi}_\eta(x)$, $[\hat{A},\hat{B}]=[\hat{b}_q^\dagger,\hat{\Psi}_\eta(x)]=\alpha_q^*(x)\hat{\Psi}_\eta(x)$, Eq.(48), we apply Eq.(A13),

$$
f(\hat{b}_q^\dagger)\hat{\Psi}_\eta(x) = \hat{\Psi}_\eta(x)f(\hat{b}_q^\dagger+\alpha_q^*(x)), \qquad (60)
$$

then

$$
\hat{\Psi}_\eta(x)f(\hat{b}_q^\dagger) = f(\hat{b}_q^\dagger-\alpha_q^*(x))\hat{\Psi}_\eta(x). \qquad (61)
$$

Therefore,

$$
\begin{aligned}
\hat{\Psi}_\eta(x)f(\hat{b}_q^\dagger)\,|N\rangle_0 &= f(\hat{b}_q^\dagger-\alpha_q^*(x))\hat{\Psi}_\eta(x)\,|N\rangle_0 \\
&= f(\hat{b}_q^\dagger-\alpha_q^*(x))\hat{F}\lambda(x)e^{-i\varphi_\eta^\dagger(x)}\,|N\rangle_0 \\
&= \hat{F}\lambda e^{-i\hat{\varphi}_\eta^\dagger(x)}f(\hat{b}_q^\dagger-\alpha_q^*(x))\,|N\rangle_0, \qquad (62)
\end{aligned}
$$

where in the last equation, we note that $\hat{\varphi}_\eta^\dagger(x)$ and $\hat{b}_q^\dagger$ commute.

Then, by setting $\hat{A}\equiv\hat{b}_q^\dagger$ and $\hat{B}\equiv i\hat{\varphi}_\eta(x)$, the commutator

$$
\begin{aligned}
[\hat{A},\hat{B}] &= [\hat{b}_q^\dagger, i\hat{\varphi}_\eta(x)] \\
&= -i\sum_{q'>0}\frac{1}{\sqrt{n_{q'}}}e^{-iq'x}[\hat{b}_q^\dagger,\hat{b}_{q'}] \\
&= -i\sum_{q'>0}\frac{1}{\sqrt{n_{q'}}}e^{-iq'x}(-\delta_{q,q'}) \\
&= -\frac{-i}{\sqrt{n_q}}e^{-iqx} = -\alpha_q^*(x), \qquad (63)
\end{aligned}
$$

is a c-number. Then we apply Eq.(A20) and obtain

$$
f(\hat{b}_q^\dagger-\alpha_q^*(x)) = e^{-i\hat{\varphi}_\eta(x)}f(\hat{b}_q^\dagger)e^{i\hat{\varphi}_\eta(x)}. \qquad (64)
$$

By putting this in Eq.(62),

$$
\begin{aligned}
\hat{\Psi}_\eta(x)f(\hat{b}_q^\dagger)\,|N\rangle_0 &= \hat{F}\lambda e^{-i\varphi_\eta^\dagger(x)}e^{-i\hat{\varphi}_\eta(x)}f(\hat{b}_q^\dagger)e^{i\hat{\varphi}_\eta(x)}\,|N\rangle_0 \\
&= \hat{F}\lambda e^{-i\hat{\varphi}_\eta^\dagger(x)}e^{-i\hat{\varphi}_\eta(x)}f(\hat{b}_q^\dagger)\,|N\rangle_0\,, \qquad (65)
\end{aligned}
$$

where we used $\hat{\varphi}_\eta(x)\,|N\rangle_0=0$. It can be seen that all these arguments are valid for more general function $f(\{\hat{b}_q^\dagger\})$. Then with $|N\rangle_f\equiv f(\{\hat{b}_q^\dagger\})\,|N\rangle_0$,

$$
\hat{\Psi}_\eta(x)\,|N\rangle_f = \hat{F}\lambda e^{-i\hat{\varphi}_\eta^\dagger(x)}e^{-i\hat{\varphi}_\eta(x)}\,|N\rangle_f\,. \qquad (66)
$$

Finally, by setting $\hat{A}\equiv -i\hat{\varphi}_\eta^\dagger(x)$ and $\hat{B}\equiv -i\hat{\varphi}_\eta(x)$, the commutator

$$
\begin{aligned}
[\hat{A},\hat{B}] &= -\left[\hat{\varphi}_\eta^\dagger(x),\hat{\varphi}_\eta(x)\right] \\
&= -\ln\left[1-e^{-i\frac{2\pi}{L}(-ia)}\right] \\
&= -\ln\left[1-e^{-\frac{2\pi a}{L}}\right] \\
&\to -\ln\left[\frac{2\pi a}{L}\right], \qquad (67)
\end{aligned}
$$

for $a\to 0^+$, is a c-number. Then using Eq. (A25),

$$
\begin{aligned}
e^{-i\hat{\varphi}_\eta^\dagger(x)}e^{-i\hat{\varphi}_\eta(x)} &= e^{-i(\hat{\varphi}_\eta^\dagger(x)+\hat{\varphi}_\eta^\dagger)}e^{-\ln[\frac{2\pi a}{L}]/2} \\
&= e^{-i\hat{\Phi}_\eta(x)}\sqrt{\frac{L}{2\pi a}}. \qquad (68)
\end{aligned}
$$

Therefore, we have the final bosonic representation of the field operator

$$
\begin{aligned}
\hat{\Psi}_\eta(x) &= \hat{F}\hat{\lambda}e^{-i\varphi_\eta^\dagger(x))}e^{-i\varphi_\eta(x)} \\
&= \hat{F}\frac{1}{\sqrt{L}}e^{-ik_{0N}x}e^{-i\Phi_\eta(x)}\sqrt{\frac{L}{2\pi a}} \\
&= \frac{1}{\sqrt{2\pi a}}\hat{F}e^{-i\tilde{\Phi}_\eta(x)}, \qquad (69)
\end{aligned}
$$

where we defined $\tilde{\Phi}_\eta(x)=\hat{\Phi}_\eta(x)+k_{0N}x$. Similarly,

$$
\hat{\Psi}_\eta^\dagger(x) = \frac{1}{\sqrt{2\pi a}}\hat{F}^\dagger e^{i\tilde{\Phi}_\eta(x)}. \qquad (70)
$$

The real field operator $\hat{\Phi}_\eta(x)$ is called a phase operator.

V. BOSONIZATION OF HAMILTONIAN WITH LINEAR DISPERSION

The discussions in the previous sections only assumed monotonic properties of the energy ϵ_k as a function of the wave number $k\in[-\infty,\infty]$. In the following, we fix the energy dispersion to be linear, $\epsilon_k=\hbar v_F k$, where Fermi velocity v_F characterizes the dispersion at the Fermi point. Then the free Hamiltonian without the interaction is

$$\hat{H}_{\rm kin} \equiv \hbar v_F \sum_{k=-\infty}^{\infty} k \,{}^*_*\, \hat{c}_k^\dagger \hat{c}_k \,{}^*_*$$
$$= \hbar v_F \int_{-L/2}^{L/2} dx \,{}^*_*\, \hat{\Psi}_\eta^\dagger(x) i\partial_x \hat{\Psi}_\eta(x) \,{}^*_*\, . \quad (71)$$

It is clear that the following commutation relation holds:

$$[\hat{H}_{\rm kin}, \hat{N}] = 0. \quad (72)$$

The ground state energy of N electron Hilbert space $\mathcal{H}_N$ is

$$E_0^N \equiv {}_0\langle N| \hat{H}_{\rm kin} |N\rangle_0 , \quad (73)$$

which is for $N > 0$, $\hbar v_F \frac{2\pi}{L} \sum_{n=1}^{N}(n - \frac{\delta_b}{2})$, for $N < 0$, $\hbar v_F \frac{2\pi}{L} \sum_{n=N+1}^{0}(-1)(n - \frac{\delta_b}{2})$ and zero for $N = 0$. Therefore,

$$E_0^N = \hbar v_F \frac{2\pi}{L} \frac{N(N+1-\delta_b)}{2}. \quad (74)$$

Let us try to derive equivalent expression of $\hat{H}_{\rm kin}$ by bosonic operators. First we study the commutation relation:

$$[\hat{H}_{\rm kin}, \hat{b}_q^\dagger]$$
$$= \hbar v_F \sum_k k \frac{i}{\sqrt{n_q}} \sum_{k'} {}^*_*\, [\hat{c}_k^\dagger \hat{c}_k, \hat{c}_{k'+q}^\dagger \hat{c}'_k]$$
$$= \hbar v_F \frac{i}{\sqrt{n_q}} \sum_{k,k'} k \,{}^*_*\, \left(\hat{c}_k^\dagger\{\hat{c}_k, \hat{c}_{k'+q}\}\hat{c}_{k'} - \hat{c}_{k'+q}^\dagger\{\hat{c}_k^\dagger, \hat{c}_{k'}\}\hat{c}_k\right) {}^*_*$$
$$= \hbar v_F \frac{i}{\sqrt{n_q}} \sum_{k,k'} k \,{}^*_*\, \left(\delta_{k,k'+q}\hat{c}_k^\dagger \hat{c}_{k'} - \delta_{k,k'}\hat{c}_{k'+q}^\dagger \hat{c}_k\right) {}^*_*$$
$$= \hbar v_F \frac{i}{\sqrt{n_q}} \sum_{k} k \,{}^*_*\, (\hat{c}_k^\dagger \hat{c}_{k-q} - \hat{c}_{k+q}^\dagger \hat{c}_k) \,{}^*_*$$
$$= \hbar v_F \frac{i}{\sqrt{n_q}} \sum_{k} \{(k+q) - k\} \hat{c}_{k+q}^\dagger \hat{c}_k$$
$$= \hbar v_F q \hat{b}_q^\dagger. \quad (75)$$

Then, if we consider any energy eigen-state of $\hat{H}_{\rm kin}$,

$$\hat{H}_{\rm kin} |E\rangle = E |E\rangle , \quad (76)$$

The state $\hat{b}_q^\dagger |E\rangle$ is also the energy eigen-state, since

$$\hat{H}_{\rm kin}\hat{b}_q^\dagger |E\rangle = \{\hat{b}_q^\dagger \hat{H}_{\rm kin} + \hbar v_F q \hat{b}_q^\dagger\} |E\rangle$$
$$= (E + \hbar v_F q)\hat{b}_q^\dagger |E\rangle , \quad (77)$$

where we used the commutation relation Eq.(75).

Since any state $|N\rangle_f$ in $\mathcal{H}_N$ can be generated by $f(\hat{b}_q^\dagger)$, $\hat{H}_{\rm kin}$ can be written with bosonic operator with the operator with $\hat{N}$ representing ground state energy:

$$\hat{H}_{\rm kin}^B = \sum_{q>0} \hbar v_F q \hat{b}_q^\dagger \hat{b}_q + \hbar v_F \frac{2\pi}{L} \frac{\hat{N}(\hat{N}+1-\delta_b)}{2}. \quad (78)$$

We can check the equivalence by examining

$$[\hat{H}_{\rm kin}^B, \hat{b}_q^\dagger] = \sum_{q'>0} \hbar v_F q' [\hat{b}_{q'}^\dagger \hat{b}_{q'}, \hat{b}_q^\dagger]$$
$$= \sum_{q'>0} \hbar v_F q' \left(\hat{b}_{q'}^\dagger [\hat{b}_{q'}, \hat{b}_q^\dagger] + [\hat{b}_{q'}^\dagger, \hat{b}_q^\dagger]\hat{b}_{q'}\right)$$
$$= \hbar v_F q \hat{b}_q^\dagger. \quad (79)$$

We introduce bosonic normal order,

$${}^*_*\, \hat{A}\hat{B}\cdots {}^*_* \equiv \hat{A}\hat{B}\cdots - {}_0\langle 0| \hat{A}\hat{B}\cdots |0\rangle_0 , \quad (80)$$

where $\hat{A}, \hat{B}, \ldots$ are bosonic creation and annihilation operators, $\{\hat{b}_q, \hat{b}_q^\dagger\}$. Then we evaluate following integral with using the relation Eq.(35),

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx}{2\pi} \,{}^*_*\, \left(\partial_x \hat{\Phi}_\eta(x)\right)^2 {}^*_*$$
$$= \left(\frac{2\pi}{L}\right)^2 \sum_{q>0} i\sqrt{n_q} \sum_{q'>0} i\sqrt{n_{q'}} e^{-\frac{a(q+q')}{2}}$$
$$\times \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx}{2\pi} \,{}^*_*\, \Big\{e^{-i(q+q')x}\hat{b}_q\hat{b}_{q'} - e^{-i(q-q')x}\hat{b}_q\hat{b}_{q'}^\dagger$$
$$-e^{-i(q'-q)x}\hat{b}_q^\dagger\hat{b}_{q'} + e^{i(q+q')x}\hat{b}_q^\dagger\hat{b}_{q'}^\dagger\Big\} {}^*_*$$
$$= \left(\frac{2\pi}{L}\right)^2 \sum_{q,q'>0} \sqrt{n_q n_{q'}} e^{-\frac{a(q+q')}{2}}$$
$$\times \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx}{2\pi} \left\{e^{i(q'-q)x}\hat{b}_{q'}^\dagger\hat{b}_q + e^{i(q-q')x}\hat{b}_q^\dagger\hat{b}_{q'}\right\}$$
$$= \frac{2\pi}{L} \sum_{q>0} n_q e^{-aq} 2\hat{b}_q^\dagger \hat{b}_q = 2\sum_{q>0} q e^{-qa} \hat{b}_q^\dagger \hat{b}_q, \quad (81)$$

where we used $\int dx \exp[\pm i(q+q')x] = 0$ since $q + q' > 0$. Therefore,

$$\hat{H}_{\rm kin}^B \quad (82)$$
$$= \hbar v_F \left[\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx}{2\pi} \frac{1}{2} \,{}^*_*\, \left(\partial_x \hat{\Phi}_\eta(x)\right)^2 {}^*_* + \frac{2\pi}{L} \frac{\hat{N}(\hat{N}+1-\delta_b)}{2}\right]$$
$$\to 2\pi\hbar v_F \int_{-\infty}^{\infty} dx \frac{1}{2} \,{}^*_*\, \hat{\rho}(x)^2 \,{}^*_*, \quad (83)$$

where the limit $L \to \infty$ is taken, and it reduces to the familiar form of the bosonized Hamiltonian in the literature. This shows that the density modulation $\hat{\rho}(x)$ costs extra kinetic energy.

VI. TOMONAGA-LUTTINGER LIQUID

A. Free part

We first discuss spinless one-dimensional fermion system with the energy dispersion

$$\epsilon_k = \frac{\hbar^2}{2m}(k^2 - k_F^2), \quad (84)$$

where the corresponding Fermionic field is

$$\begin{aligned}\hat{\Psi}(x) &= \frac{1}{\sqrt{L}}\sum_{k=-\infty}^{\infty} e^{ikx}\hat{c}_k \\ &= \frac{1}{\sqrt{L}}\sum_{k=0}^{\infty}\{e^{-ikx}\hat{c}_{-k}+e^{ikx}\hat{c}_k\} \quad (85)\\ &= \frac{1}{\sqrt{L}}\sum_{k=-k_F}^{\infty}\{e^{-i(k+k_F)x}\hat{c}_{kL}+e^{i(k+k_F)x}\hat{c}_{kR}\},\end{aligned}$$

where the first term is left-going wave and the second term is right-going wave. We have shifted the wave number relative to the Fermi wave number k_F and defined $\hat{c}_{k\nu} \equiv \hat{c}_{\mp(k_F+k)}$ for $\nu = L/R$. We also define the dispersion relation for the mode ν, $\epsilon_{k\nu} = \epsilon(\mp(k+k_F)) = \frac{\hbar^2}{2m}\left((k+k_F)^2 - k_F^2\right) = \frac{\hbar^2}{m}(k_F k + \frac{k^2}{2})$ for $k > -k_F$ and $\epsilon_{k\nu} = -\epsilon(0) + \hbar v_F(k+k_F)$ for $k < -k_F$, where $\epsilon(0) \equiv \frac{\hbar^2 k_F^2}{2m}$ which satisfies the condition of monotonic dispersion. (*Approximation I*) Now the Fermionic field operator is made of two terms

$$\hat{\Psi}^{\mathrm{TL}}(x) = e^{-ik_Fx}\hat{\Psi}_L(x) + e^{ik_Fx}\hat{\Psi}_R(x), \quad (86)$$

$$\hat{\Psi}_\nu(x) \equiv \frac{1}{\sqrt{L}}\sum_{k=-\infty}^{\infty} e^{\mp ikx}\hat{c}_{k\nu}, \quad (87)$$

for $\nu = L$ and R.

Repeating the procedure in the previous sections for R mode, where the sign of k and ∂_x is opposite and we have the expressions for bosonic field operator

$$\hat{\Phi}_\nu(x) = -\sum_{q>0}\frac{1}{\sqrt{n_q}}e^{-\frac{aq}{2}}[e^{\mp iqx}\hat{b}_{q\nu} + e^{\pm iqx}\hat{b}^\dagger_{q\nu}], \quad (88)$$

where $q = 2\pi n_q/L$ and upper/lower sign represents L/R mode, which satisfies commutation relation

$$[\hat{\Phi}_\nu(x), \hat{\Phi}_{\nu'}(x')] = -\pi i\delta_{\nu\nu'}\mathrm{Sgn}(x-x')s_\nu, \quad (89)$$

where $s_L = 1$ and $s_R = -1$. Then the Bosonized Fermionic field operator for $L \to \infty$ is

$$\hat{\Psi}^B_\nu(x) \sim \frac{1}{\sqrt{2\pi a}}\hat{F}_\nu e^{-i\hat{\Phi}_\nu(x)}, \quad (90)$$

and the particle density operators are

$$\hat{\rho}_\nu(x) = \pm\frac{1}{2\pi}\partial_x\hat{\Phi}_\nu(x) + \frac{1}{L}\hat{N}_\nu. \quad (91)$$

If we are only interested in the low energy physics much smaller than $\epsilon(0) = \frac{\hbar^2 k_F^2}{2m} = \epsilon_F$, we are justified to linearlize the dispersion $\epsilon_{k\nu} \sim \hbar v_F k$ where $v_F = \hbar k_F/m$. (*Approximation II*) Then the free Hamiltonian is

$$\begin{aligned}\hat{H}_{\mathrm{kin}} &= \hbar v_F\int_{-\frac{L}{2}}^{\frac{L}{2}} dx \, {}^*_*[\hat{\Psi}^\dagger_L(x)i\partial_x\hat{\Psi}_L(x) \\ &\quad + \hat{\Psi}^\dagger_R(x)(-i\partial_x)\hat{\Psi}_R(x)]{}^*_* \\ &\to \hbar v_F\sum_\nu\Big[\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dx}{2\pi}\frac{1}{2}\,{}^*_*\Big(\partial_x\hat{\Phi}_\nu(x)\Big)^2{}^*_* \\ &\quad + \frac{2\pi}{L}\frac{\hat{N}_\nu(\hat{N}_\nu+1-\delta_b)}{2}\Big] \\ &\to 2\pi\hbar v_F\int dx\frac{1}{2}\,{}^*_*\left[\hat{\rho}_L^2(x)+\hat{\rho}_R^2(x)\right]{}^*_*, \quad (92)\end{aligned}$$

where in the last expression we took the limit $L \to \infty$.

B. Interaction part: spin-less case

There are two kinds of interaction process,

- Inter-mode scattering

  This process is with transferred momentum p ($|p| \ll k_F$), the electron k' in the R mode is scattered to $k'+p$ and at the same time, the electron k in the L mode is scattered to $k-p$, like $\hat{c}^\dagger_{k-pL}\hat{c}_{kL}\hat{c}^\dagger_{k'+pR}\hat{c}_{k'R}$. The amplitude of the scattering is defined as $g_2(p)$. For simplicity, we neglect p dependence of the coupling strength and this term is expressed by $g_2\int dx\,{}^*_*\hat{\rho}_L(x)\hat{\rho}_R(x){}^*_*$.

- Intra-mode scattering

  This process is that for each mode ν, the electron k' is scattered to $k'+p$ and at the same time, the other electron of the same mode k is scattered to $k-p$, like $\hat{c}^\dagger_{k-q\nu}\hat{c}_{k\nu}\hat{c}^\dagger_{k'+q\nu}\hat{c}_{k'\nu}$. The amplitude of the scattering is defined as $g_4(p)$, while we neglect p dependence. This term is expressed by $\frac{1}{2}g_4\int dx\,{}^*_*\hat{\rho}^2_\nu(x){}^*_*$.

Therefore, the interaction Hamiltonian is

$$\hat{H}_{\mathrm{int}} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\,{}^*_*\left[g_2\hat{\rho}_L(x)\hat{\rho}_R(x) + \frac{1}{2}g_4(\hat{\rho}^2_L(x)+\hat{\rho}^2_R(x))\right]{}^*_* \quad (93)$$

C. Diagonalization by Bogoliubov transformation

The total Hamiltonian is

$$\hat{H}_0 \equiv \hat{H}_{\mathrm{kin}} + \hat{H}_{\mathrm{int}}, \quad (94)$$

which can be diagonalized with so-called Bogoliubov transformation. Let us define new density operators as

$$\hat{\rho}_+(x) \equiv \frac{1}{\sqrt{2}}\left(\hat{\rho}_R(x)+\hat{\rho}_L(x)\right), \quad (95)$$

$$\hat{\rho}_-(x) \equiv \frac{1}{\sqrt{2}}\left(\hat{\rho}_R(x)-\hat{\rho}_L(x)\right), \quad (96)$$

where $\hat{\rho}_+$ represents average density fluctuations and $\hat{\rho}_-$ represents 'current' fluctuation. Then the total Hamiltonian is

$$\begin{aligned}\hat{H}_0 &= \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \,{}^*_*\Big[\frac{2\pi\hbar v_F + g_4}{2}\left(\hat{\rho}_L^2(x) + \hat{\rho}_R^2(x)\right) \\ &\quad + g_2\hat{\rho}_L(x)\hat{\rho}_R(x)\Big]{}^*_* \\ &= \frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx \,{}^*_*\Big[(2\pi\hbar v_F + g_4 + g_2)\hat{\rho}_+^2(x) \\ &\quad + (2\pi\hbar v_F + g_4 - g_2)\hat{\rho}_-^2(x)\Big]{}^*_* \\ &= \pi\hbar v \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \,{}^*_*\left[\frac{1}{K}\hat{\rho}_+^2(x) + K\hat{\rho}_-^2(x)\right]{}^*_*, \end{aligned} \tag{97}$$

where we defined renormalized velocity v and the interaction parameter K by

$$v \equiv \sqrt{\left(v_F + \frac{g_4}{2\pi\hbar}\right)^2 - \left(\frac{g_2}{2\pi\hbar}\right)^2}, \tag{98}$$

$$K \equiv \sqrt{\frac{2\pi\hbar v_F + g_4 - g_2}{2\pi\hbar v_F + g_4 + g_2}}. \tag{99}$$

As can be seen easily, for non-interacting system, $v = v_F$ and $K = 1$. And in general cases with repulsive interaction $g_2, g_4 \geq 0$, $K \leq 1$. It should be noted that the density operators $\hat{\rho}_\pm$ is made of linear combination of $\hat{b}_\nu^\dagger$, $\hat{b}_\nu$, $\hat{H}_0$ is the operator binary in $\hat{b}_\nu^\dagger$, $\hat{b}_\nu$. Therefore, we can diagonalize it and exactly solvable even with finite interaction!

We will do this explicitly in the following. Rewriting the Hamiltonian with the phase operator,

$$\begin{aligned}\hat{H}_0 &= \pi\hbar v \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \,{}^*_*\Big[\frac{1}{K}\frac{1}{2}\left(\hat{\rho}_R(x) + \hat{\rho}_L(x)\right)^2 \\ &\quad + K\frac{1}{2}\left(\hat{\rho}_R(x) - \hat{\rho}_L(x)\right)^2\Big]{}^*_* \\ &= \frac{\pi\hbar v}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx \,{}^*_*\Big[\left(\frac{1}{K} + K\right)\left(\hat{\rho}_R^2(x) + \hat{\rho}_L^2(x)\right) \\ &\quad + 2\left(\frac{1}{K} - K\right)\hat{\rho}_L(x)\hat{\rho}_R(x)\Big]{}^*_*. \end{aligned} \tag{100}$$

Since

$$\begin{aligned}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx \,{}^*_*\hat{\rho}_\nu^2(x){}^*_* &= \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \,{}^*_*\left\{\pm\frac{1}{2\pi}\partial_x\hat{\Phi}_\nu(x)\right\}^2{}^*_* \\ &= \frac{2}{2\pi}\sum_{q>0} q\hat{b}_{q\nu}^\dagger\hat{b}_{q\nu}, \end{aligned} \tag{101}$$

where we used Eq.(81) and noting

$$\begin{aligned}\partial_x\hat{\Phi}_R(x) &= -\sum_{q>0}\frac{1}{\sqrt{n_q}}e^{-\frac{aq}{2}}[iqe^{iqx}\hat{b}_{qR} - iqe^{-iqx}\hat{b}_{qR}^\dagger] \\ &= -\frac{2\pi}{L}\sum_{q>0} i\sqrt{n_q}(e^{iqx}\hat{b}_{qR} - e^{-iqx}\hat{b}_{qR}^\dagger)e^{-\frac{aq}{2}}, \end{aligned} \tag{102}$$

we have

$$\begin{aligned}&\int_{-\frac{L}{2}}^{\frac{L}{2}} dx \,{}^*_*\hat{\rho}_L(x)\hat{\rho}_R(x){}^*_* \\ &= -\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx}{4\pi^2}\,{}^*_*\{\partial_x\hat{\Phi}_L(x)\partial_x\hat{\Phi}_R(x)\}{}^*_* \\ &= -\left(\frac{2\pi}{L}\right)^2\sum_{q,q'>0}\sqrt{n_q n_{q'}}e^{-\frac{a(q+q')}{2}} \\ &\times \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx}{4\pi^2}\,{}^*_*(e^{-iqx}\hat{b}_{qL} - e^{iqx}\hat{b}_{qL}^\dagger)(e^{iq'x}\hat{b}_{q'R} - e^{-iq'x}\hat{b}_{q'R}^\dagger){}^*_* \\ &= -\left(\frac{2\pi}{L}\right)^2\sum_{q,q'>0}\sqrt{n_q n_{q'}}e^{-\frac{a(q+q')}{2}} \\ &\times \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx}{4\pi^2}\,{}^*_*\left\{e^{i(q'-q)x}\hat{b}_{qL}\hat{b}_{q'R} + e^{i(q-q')x}\hat{b}_{qL}^\dagger\hat{b}_{q'R}^\dagger\right\}{}^*_* \\ &= -\left(\frac{2\pi}{L}\right)^2\sum_{q>0} n_q e^{-aq}\frac{L}{4\pi^2}\,{}^*_*\left(\hat{b}_{qL}\hat{b}_{qR} + \hat{b}_{qL}^\dagger\hat{b}_{qR}^\dagger\right){}^*_* \\ &= -\frac{1}{2\pi}\sum_{q>0} q\,{}^*_*\left(\hat{b}_{qL}\hat{b}_{qR} + \hat{b}_{qL}^\dagger\hat{b}_{qR}^\dagger\right){}^*_*. \end{aligned} \tag{103}$$

Therefore,

$$\begin{aligned}\hat{H}_0 &= \frac{\pi\hbar v}{2}\Big\{\left(\frac{1}{K} + K\right)\frac{2}{2\pi}\sum_{q>0} q\left(\hat{b}_{qL}^\dagger\hat{b}_{qL} + \hat{b}_{qR}^\dagger\hat{b}_{qR}\right) \\ &\quad - \left(\frac{1}{K} - K\right)\frac{2}{2\pi}\sum_{q>0} q\,{}^*_*\left(\hat{b}_{qL}\hat{b}_{qR} + \hat{b}_{qL}^\dagger\hat{b}_{qR}^\dagger\right){}^*_*\Big\} \\ &= \frac{1}{2}\sum_{q>0}\hbar v q\,{}^*_*\Big\{\left(\frac{1}{K} + K\right)(\hat{b}_{qL}^\dagger\hat{b}_{qL} + \hat{b}_{qR}^\dagger\hat{b}_{qR}) \\ &\quad - \left(\frac{1}{K} - K\right)(\hat{b}_{qL}\hat{b}_{qR} + \hat{b}_{qL}^\dagger\hat{b}_{qR}^\dagger)\Big\}{}^*_*, \end{aligned} \tag{104}$$

where $[\hat{b}_{q\nu}, \hat{b}_{q'\nu'}^\dagger] = \delta_{qq'}\delta_{\nu\nu'}$, and $[\hat{b}_{q\nu}, \hat{b}_{q'\nu'}] = [\hat{b}_{q\nu}^\dagger, \hat{b}_{q'\nu'}^\dagger] = 0$. We then introduce

$$\hat{b}_{q+} \equiv \frac{1}{\sqrt{2}}(\hat{b}_{qL} + \hat{b}_{qR}), \tag{105}$$

$$\hat{b}_{q-} \equiv \frac{1}{\sqrt{2}}(\hat{b}_{qL} - \hat{b}_{qR}), \tag{106}$$

and $\hat{b}_{q\pm}^\dagger$ are also defined. Clearly, $[\hat{b}_{qu}, \hat{b}_{q'u'}] = [\hat{b}_{qu}^\dagger, \hat{b}_{q'u'}^\dagger] = 0$ and $[\hat{b}_{qu}, \hat{b}_{q'u'}^\dagger] = \delta_{qq'}\delta_{uu'}$ where u, u' is $+/-$. Then we can show that

$$\hat{b}_{qL}^\dagger\hat{b}_{qL} + \hat{b}_{qR}^\dagger\hat{b}_{qR} = \hat{b}_{q+}^\dagger\hat{b}_{q+} + \hat{b}_{q-}^\dagger\hat{b}_{q-}, \tag{107}$$

$$\hat{b}_{qL}\hat{b}_{qR} + \hat{b}_{qL}^\dagger\hat{b}_{qR}^\dagger = \frac{1}{2}\left(\hat{b}_{q+}^2 - \hat{b}_{q-}^2 + \hat{b}_{q+}^{\dagger 2} - \hat{b}_{q-}^{\dagger 2}\right) \tag{108}$$

Then

$$\hat{H}_0 = \frac{1}{2}\sum_{q>0}\hbar vq \times {}^*_*\ \Big[\left\{\left(\frac{1}{K}+K\right)\hat{b}^\dagger_{q+}\hat{b}_{q+} - \frac{1}{2}\left(\frac{1}{K}-K\right)(\hat{b}^2_{q+}+\hat{b}^{\dagger 2}_{q+})\right\} + \left\{\left(\frac{1}{K}+K\right)\hat{b}^\dagger_{q-}\hat{b}_{q-} + \frac{1}{2}\left(\frac{1}{K}-K\right)(\hat{b}^2_{q-}+\hat{b}^{\dagger 2}_{q-})\right\}\Big]\ {}^*_* . \tag{109}$$

Each $+/-$ part can be diagonalized by Bogoliubov transformation,

$$\hat{b} = u\hat{a} + v\hat{a}^\dagger, \tag{110}$$
$$\hat{b}^\dagger = u\hat{a}^\dagger + v\hat{a}, \tag{111}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are new bosonic operators that satisfy $[\hat{a},\hat{a}^\dagger]=1$, u and v are (in general complex but here we set real) constants that satisfy $u^2-v^2=1$. This last property guarantees the commutation relation $[\hat{b},\hat{b}^\dagger]=1$. Now, the Hamiltonian with real parameters A, B (with $A>|B|$),

$$\hat{h} = {}^*_*\left\{Ab^\dagger b + \frac{1}{2}B(b^2+b^{\dagger 2})\right\}{}^*_*, \tag{112}$$

is rewritten with the operators $a, a^\dagger$,

$$\begin{aligned}\hat{h} &= A\,{}^*_*\,(u\hat{a}^\dagger+v\hat{a})(u\hat{a}+v\hat{a}^\dagger)\,{}^*_* \\ &\quad + \frac{B}{2}\,{}^*_*\left\{(u\hat{a}^\dagger+v\hat{a})^2+(u\hat{a}+v\hat{a}^\dagger)^2\right\}{}^*_* \\ &= \left[A(u^2+v^2)+2Buv\right]\hat{a}^\dagger\hat{a} \\ &\quad + \left[Auv+\frac{B}{2}(u^2+v^2)\right](\hat{a}^\dagger\hat{a}^\dagger+\hat{a}\hat{a}). \end{aligned} \tag{113}$$

Then we require the last term to vanish, by the condition

$$Auv + \frac{B}{2}(u^2+v^2) = 0. \tag{114}$$

Using $v^2=u^2-1$, we get the equation for u^2,

$$u^2(u^2-1) - \frac{\alpha}{4} = 0, \tag{115}$$

where $\alpha \equiv B^2/(A^2-B^2) > 0$. From the solutions

$$u^2 = \frac{1}{2}(1\pm\sqrt{1+\alpha}), \tag{116}$$

we take plus sign since $\alpha > 0$. Then $u = \sqrt{\left(\sqrt{1+a}+1\right)/2}$. The sign of $v=\pm\sqrt{u^2-1}$ is determined to achieve the condition, Eq.(114). Hence,

$$u = \sqrt{\frac{1}{2}\left\{\sqrt{\frac{A^2}{A^2-B^2}}+1\right\}}, \tag{117}$$

$$v = -\frac{B}{|B|}\sqrt{\frac{1}{2}\left\{\sqrt{\frac{A^2}{A^2-B^2}}-1\right\}}. \tag{118}$$

Then the coefficient of diagonal part is

$$A(u^2+v^2)+2Buv = \sqrt{A^2-B^2} \tag{119}$$

and the Hamiltonian is diagonalized

$$\hat{h} = \sqrt{A^2-B^2}\,\hat{a}^\dagger\hat{a}. \tag{120}$$

With putting $A=K^{-1}+K$ and $B=\mp(K^{-1}-K)$, we have

$$\sqrt{A^2-B^2} = 2, \tag{121}$$
$$u = \frac{1}{2}\left\{\frac{1}{\sqrt{K}}+\sqrt{K}\right\}, \tag{122}$$
$$v = \pm\frac{1}{2}\left\{\frac{1}{\sqrt{K}}-\sqrt{K}\right\}, \tag{123}$$

then the Hamiltonian reduces to

$$\hat{H}_0 = \sum_{q>0}\hbar vq\left\{\hat{a}^\dagger_{q+}\hat{a}_{q+}+\hat{a}^\dagger_{q-}\hat{a}_{q-}\right\}. \tag{124}$$

Therefore, the Hamiltonian is expressed with the sum of independent bosons (Harmonic oscillators) Hamiltonian.

Let us rewrite the total Hamiltonian in the standard form. We introduce

$$\hat{\theta}(x) = \frac{1}{2}[\hat{\Phi}_R(x)-\hat{\Phi}_L(x)], \tag{125}$$
$$\hat{\phi}(x) = \frac{1}{2}[\hat{\Phi}_R(x)+\hat{\Phi}_L(x)], \tag{126}$$

then the commutation relation reads

$$\begin{aligned}[\hat{\theta}(x),\hat{\phi}(x')] &= \frac{1}{4}\left\{[\hat{\Phi}_R(x),\hat{\Phi}_R(x')]-[\hat{\Phi}_L(x),\hat{\Phi}_L(x')]\right\} \\ &= \frac{1}{4}\{i\pi\mathrm{Sgn}(x-x')+i\pi\mathrm{Sgn}(x-x')\} \\ &= \frac{\pi i}{2}\mathrm{Sgn}(x-x'). \end{aligned} \tag{127}$$

Moreover,

$$\begin{aligned}\partial_x\hat{\theta}(x) &= \frac{1}{2}[\partial_x\hat{\Phi}_R(x)-\partial_x\hat{\Phi}_L(x)] \\ &= \frac{1}{2}[-2\pi\hat{\rho}_R(x)-2\pi\hat{\rho}_L(x)] \\ &= -\pi[\hat{\rho}_R(x)+\hat{\rho}_L(x)] \\ &= -\sqrt{2}\pi\hat{\rho}_+(x), \end{aligned} \tag{128}$$
$$\partial_x\hat{\phi}(x) = -\sqrt{2}\pi\hat{\rho}_-(x). \tag{129}$$

The total charge $\hat{\rho}_{\mathrm{T}}(x) \equiv \hat{\rho}_L(x)+\hat{\rho}_R(x)$ is given by

$$\hat{\rho}_{\mathrm{T}}(x) = -\frac{1}{\pi}\partial_x\hat{\theta}(x). \tag{130}$$

Therefore, we get the total Hamiltonian expressed with the phase operators,

$$\hat{H}_0 = \hbar v\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dx}{2\pi}\,{}^*_*\left[\frac{1}{K}(\partial_x\hat{\theta}(x))^2+K(\partial_x\hat{\phi}(x))^2\right]{}^*_* . \tag{131}$$

Corresponding Fermionic field operator is given by

$$\begin{aligned}\Psi(x) &= \frac{1}{\sqrt{2\pi a}}\sum_\nu e^{\mp ik_F x}\hat{F}_\nu e^{-i\hat{\Phi}_\nu(x)} \\ &= \frac{1}{\sqrt{2\pi a}}\sum_\nu \hat{F}_\nu e^{\mp i(k_F x-\hat{\theta}(x))-i\hat{\phi}(x)}. \qquad (132)\end{aligned}$$

D. Fermion with spin

We then discuss Fermionic particles with spin 1/2. Then we have the modes η's represented with $\{\nu, s\}$, where $\nu = L/S$ and $s =\uparrow / \downarrow$. For simplicity, we do not discuss the effect of external magnetic field (Zeeman term). Four density fluctuation operators $\hat{\rho}_{L\uparrow}(x), \hat{\rho}_{L\downarrow}(x), \hat{\rho}_{R\uparrow}(x), \hat{\rho}_{R\downarrow}(x)$ obey following Hamiltonian:

- Kinetic term

  Kinetic energy is just the sum of each contribution of spins,

$$\hat{H}_{\mathrm{kin}} = \sum_s 2\pi\hbar v_F \int dx \frac{1}{2} \overset{*}{*} \left[\hat{\rho}_{Ls}^2 + \hat{\rho}_{Rs}^2\right] \overset{*}{*}. \qquad (133)$$

- Inter-mode scattering

  There are two types of inter-mode scattering, $g_{2\parallel}$ and $g_{2\perp}$, and hence

$$\hat{H}_{\mathrm{inter}} = \sum_s \int dx \overset{*}{*} \left[g_{2\parallel}\hat{\rho}_{Ls}\hat{\rho}_{Rs} + g_{2\perp}\hat{\rho}_{Ls}\hat{\rho}_{R\bar{s}}\right] \overset{*}{*} \qquad (134)$$

  where $\bar{s}$ is opposite of s.

- Intra-mode scattering

  There are two types of intra-mode scattering, $g_{4\parallel}$ and $g_{4\perp}$, and hence

$$\begin{aligned}\hat{H}_{\mathrm{intra}} &= \sum_s \int dx \overset{*}{*} \Big[\frac{1}{2}g_{4\parallel}\left(\hat{\rho}_{Ls}^2 + \hat{\rho}_{Rs}^2\right) \\ &\quad +\frac{1}{2}g_{4\perp}\left(\hat{\rho}_{Ls}\hat{\rho}_{L\bar{s}} + \hat{\rho}_{Rs}\hat{\rho}_{R\bar{s}}\right)\Big] \overset{*}{*}. \qquad (135)\end{aligned}$$

- Backscattering

  For large momentum $Q \sim 2k_F$, a particle of k in L mode with spin s is scattered to $k+Q$ in R mode with spin s. At the same time, another particle of k' in R mode with spin s ($\bar{s}$) is scattered to $k'-Q$ in L mode with spin s ($\bar{s}$). The process between the same spin s has the amplitude $g_{1\parallel}$ and is represented with

$$\begin{aligned}&\hat{c}^\dagger_{L,k'-Q,s}\hat{c}_{R,k',s}\hat{c}^\dagger_{R,k+Q,s}\hat{c}_{L,k,s} \\ &= -\hat{c}^\dagger_{L,k'-Q,s}\hat{c}_{L,k,s}\hat{c}^\dagger_{R,k+Q,s}\hat{c}_{R,k',s} \\ &\to -\hat{\rho}_{Ls}\hat{\rho}_{Rs}, \qquad (136)\end{aligned}$$

  where we used anti-commutation rule. For spin-less system, this term is equivalent to g_2 process, and was absorbed to the g_2 term. The Hamiltonian is

$$\hat{H}_{\mathrm{ex}\parallel} = -\sum_s \int dx \overset{*}{*} g_{1\parallel}\hat{\rho}_{Ls}\hat{\rho}_{Rs} \overset{*}{*}. \qquad (137)$$

  The process between opposite spins has the amplitude $g_{1\perp}$ and is represented with

$$\begin{aligned}&\hat{c}^\dagger_{L,k'-Q,s}\hat{c}_{R,k',s}\hat{c}^\dagger_{R,k+Q,\bar{s}}\hat{c}_{L,k,\bar{s}} \\ &= \hat{c}^\dagger_{R,k+Q,\bar{s}}\hat{c}^\dagger_{L,k'-Q,s}\hat{c}_{R,k',s}\hat{c}_{L,k,\bar{s}}, \qquad (138)\end{aligned}$$

  which cannot be expressed by the density operators. The Hamiltonian is

$$\begin{aligned}\hat{H}_{\mathrm{ex}\perp} &= \sum_s \int dx \frac{g_{1\perp}}{2}\Big[\hat{\Psi}^\dagger_{R\bar{s}}(x)\hat{\Psi}^\dagger_{Ls}(x)\hat{\Psi}_{Rs}(x)\hat{\Psi}_{L\bar{s}}(x) \\ &\quad +\mathrm{h.c.}\Big], \qquad (139)\end{aligned}$$

  where h.c. represents Hermite conjugate. Since this term makes the Hamiltonian not diagonalizable, we set aside this term until an adequate occasion.

The total Hamiltonian, excluding $\hat{H}_{\mathrm{ex}\perp}$, is[3]

$$\begin{aligned}\hat{H}_0 &= \hat{H}_{\mathrm{kin}} + \hat{H}_{\mathrm{inter}} + \hat{H}_{\mathrm{intra}} + \hat{H}_{\mathrm{ex}\parallel} \\ &= \sum_s \int dx \overset{*}{*} \Big[(2\pi\hbar v_F)\frac{1}{2}\left(\hat{\rho}_{Ls}^2 + \hat{\rho}_{Rs}^2\right) \\ &\quad +g_{2\parallel}\hat{\rho}_{Ls}\hat{\rho}_{Rs} + g_{2\perp}\hat{\rho}_{Ls}\hat{\rho}_{R\bar{s}} \\ &\quad +\frac{1}{2}g_{4\parallel}(\hat{\rho}_{Ls}^2 + \hat{\rho}_{Rs}^2) + \frac{1}{2}g_{4\perp}(\hat{\rho}_{Ls}\hat{\rho}_{L\bar{s}} + \hat{\rho}_{Rs}\hat{\rho}_{R\bar{s}}) \\ &\quad -g_{1\parallel}\hat{\rho}_{Ls}\hat{\rho}_{Rs}\Big] \overset{*}{*}. \qquad (140)\end{aligned}$$

The following discussion assumes usual Coulombic potential occurring in the same one-dimensional subband, which satisfies $g_{4\parallel} = g_{4\perp} \equiv g_4$, $g_{2\parallel} = g_{2\perp} \equiv g_2$ and $g_{1\parallel} = g_{1\perp} \equiv g_1$. Note that this may be changed if one discusses other systems, for example, two or more edge channels closely situated but separated with a thin insulating region. In order to diagonalize the Hamiltonian, we first introduce charge ($\hat{\rho}_\nu$) and spin ($\hat{\sigma}_\nu$) density operators of mode ν,

$$\hat{\rho}_\nu \equiv \frac{1}{\sqrt{2}}(\hat{\rho}_{\nu\uparrow} + \hat{\rho}_{\nu\downarrow}), \qquad (141)$$

$$\hat{\sigma}_\nu \equiv \frac{1}{\sqrt{2}}(\hat{\rho}_{\nu\uparrow} - \hat{\rho}_{\nu\downarrow}). \qquad (142)$$

By using the inverse transformations

$$\hat{\rho}_{\nu\uparrow} = \frac{1}{\sqrt{2}}(\hat{\rho}_\nu + \hat{\sigma}_\nu), \qquad (143)$$

$$\hat{\rho}_{\nu\downarrow} = \frac{1}{\sqrt{2}}(\hat{\rho}_\nu - \hat{\sigma}_\nu). \qquad (144)$$

in the Hamiltonian, we obtain

$$\begin{aligned}\hat{H}_0 &= \int dx * \Big[\frac{2\pi\hbar v_F + 2g_4}{2}(\hat{\rho}_L^2 + \hat{\rho}_R^2) + (2g_2 - g_1)\hat{\rho}_R\hat{\rho}_L \\ &\quad + \frac{2\pi\hbar v_F}{2}(\hat{\sigma}_L^2 + \hat{\sigma}_R^2) - g_1\hat{\sigma}_R\hat{\sigma}_L\Big] * \\ &\equiv \hat{H}_\rho + \hat{H}_\sigma, \end{aligned} \quad (145)$$

where the spin and charge parts of the Hamiltonian are decoupled (spin-charge separation), which is a generic property of the one-dimensional interacting system (without Zeeman term). Following the same procedures as in the spin-less case, these Hamiltonians are diagonalized

$$\hat{H}_\rho = \pi\hbar v_\rho \int dx * \left[\frac{1}{K_\rho}\hat{\rho}_+^2 + K_\rho \hat{\rho}_-^2\right] *, \quad (146)$$

$$\hat{H}_\rho = \pi\hbar v_\sigma \int dx * \left[\frac{1}{K_\sigma}\hat{\sigma}_+^2 + K_\sigma \hat{\sigma}_-^2\right] *, \quad (147)$$

where the average charge (spin) density fluctuation '+' and charge (spin) 'current' fluctuation '−' are introduced

$$\hat{\rho}_+ = \frac{1}{\sqrt{2}}(\hat{\rho}_R + \hat{\rho}_L), \quad (148)$$

$$\hat{\rho}_- = \frac{1}{\sqrt{2}}(\hat{\rho}_R - \hat{\rho}_L), \quad (149)$$

$$\hat{\sigma}_+ = \frac{1}{\sqrt{2}}(\hat{\sigma}_R + \hat{\sigma}_L), \quad (150)$$

$$\hat{\sigma}_- = \frac{1}{\sqrt{2}}(\hat{\sigma}_R - \hat{\sigma}_L). \quad (151)$$

Effective velocities are

$$v_\rho \equiv \sqrt{\left(v_F + \frac{g_4}{\pi\hbar}\right)^2 - \left(\frac{2g_2 - g_1}{2\pi\hbar}\right)^2}, \quad (152)$$

$$v_\sigma \equiv \sqrt{v_F^2 - \left(\frac{g_1}{2\pi\hbar}\right)^2}, \quad (153)$$

and the coupling constants are

$$K_\rho \equiv \sqrt{\frac{2\pi\hbar v_F + 2g_4 - 2g_2 + g_1}{2\pi\hbar v_F + 2g_4 + 2g_2 - g_1}}, \quad (154)$$

$$K_\sigma \equiv \sqrt{\frac{2\pi\hbar v_F + g_1}{2\pi\hbar v_F - g_1}}. \quad (155)$$

In most cases, $v_\sigma < v_F < v_\rho$ and $K_\rho < 1 \le K_\sigma$ is satisfied. SU(2) symmetry requires $K_\sigma = 1$. Then, Tomonaga-Luttinger liquid is characterized with three parameters K_ρ, v_ρ, v_σ.

We introduce Boson field operators, $\hat{\Phi}_{\nu s}$, which are related to the density operators by

$$\hat{\rho}_{\nu s}(x) = \pm\frac{1}{2\pi}\partial_x \hat{\Phi}_{\nu s}(x) + \frac{1}{L}\hat{N}_{\nu s}, \quad (156)$$

which satisfy commutation relations

$$[\hat{\Phi}_{\nu s}(x), \hat{\Phi}_{\nu' s'}(x')] = -\pi i \delta_{\nu\nu'}\delta_{ss'}\mathrm{Sgn}(x - x')s_\nu, \quad (157)$$

where $s_L = 1$ and $s_R = -1$. Then we introduce charge/spin operators for mode ν,

$$\hat{\Phi}_{\nu\rho} \equiv \frac{1}{\sqrt{2}}\left[\hat{\Phi}_{\nu\uparrow} + \hat{\Phi}_{\nu\downarrow}\right], \quad (158)$$

$$\hat{\Phi}_{\nu\sigma} \equiv \frac{1}{\sqrt{2}}\left[\hat{\Phi}_{\nu\uparrow} - \hat{\Phi}_{\nu\downarrow}\right]. \quad (159)$$

These satisfy commutation relation

$$\begin{aligned}&[\hat{\Phi}_{\nu\rho}(x), \hat{\Phi}_{\nu'\rho}(x')] \\ &= \frac{1}{2}\left\{[\hat{\Phi}_{\nu\uparrow}(x), \hat{\Phi}_{\nu'\uparrow}(x')] + [\hat{\Phi}_{\nu\downarrow}(x), \hat{\Phi}_{\nu'\downarrow}(x')]\right\} \\ &= -\pi i \delta_{\nu\nu'}\mathrm{Sgn}(x - x')s_\nu. \end{aligned} \quad (160)$$

Similarly,

$$[\hat{\Phi}_{\nu\sigma}(x), \hat{\Phi}_{\nu'\sigma}(x')] = -\pi i \delta_{\nu\nu'}\mathrm{Sgn}(x - x')s_\nu. \quad (161)$$

While,

$$\begin{aligned}&[\hat{\Phi}_{\nu\rho}(x), \hat{\Phi}_{\nu'\sigma}(x')] \\ &= \frac{1}{2}\left\{[\hat{\Phi}_{\nu\uparrow}(x), \hat{\Phi}_{\nu'\uparrow}(x')] - [\hat{\Phi}_{\nu\downarrow}(x), \hat{\Phi}_{\nu'\downarrow}(x')]\right\} \\ &= 0. \end{aligned} \quad (162)$$

Therefore, we have

$$[\hat{\Phi}_{\nu\mu}(x), \hat{\Phi}_{\nu'\mu'}(x')] = -\pi i \delta_{\nu\nu'}\delta_{\mu\mu'}\mathrm{Sgn}(x - x')s_\nu, \quad (163)$$

where μ, μ' are ρ or σ.

Moreover, we introduce

$$\hat{\theta}_{\rho/\sigma} \equiv \frac{1}{2\sqrt{2}}\left[\hat{\Phi}_{R\rho/\sigma} - \hat{\Phi}_{L\rho/\sigma}\right], \quad (164)$$

$$\hat{\phi}_{\rho/\sigma} \equiv \frac{1}{2\sqrt{2}}\left[\hat{\Phi}_{R\rho/\sigma} + \hat{\Phi}_{L\rho/\sigma}\right]. \quad (165)$$

Then the commutation relations are

$$\begin{aligned}&\left[\hat{\theta}_\rho(x), \hat{\phi}_\rho(x')\right] \\ &= \frac{1}{(2\sqrt{2})^2}\left\{[\hat{\Phi}_{R\rho}(x), \hat{\Phi}_{R\rho}(x')] - [\hat{\Phi}_{L\rho}(x), \hat{\Phi}_{L\rho}(x')]\right\} \\ &= \frac{1}{8}\left\{2\pi i \mathrm{Sgn}(x - x')\right\} = \frac{i\pi}{4}\mathrm{Sgn}(x - x'), \end{aligned} \quad (166)$$

and $[\hat{\theta}_\rho(x), \hat{\theta}_\rho(x')] = [\hat{\phi}_\rho(x), \hat{\phi}_\rho(x')] = 0$. Similarly,

$$\left[\hat{\theta}_\sigma(x), \hat{\phi}_\sigma(x')\right] = \frac{i\pi}{4}\mathrm{Sgn}(x - x'), \quad (167)$$

and $[\hat{\theta}_\sigma(x), \hat{\theta}_\sigma(x')] = [\hat{\phi}_\sigma(x), \hat{\phi}_\sigma(x')] = 0$. Finally, noting the relation Eq.(162), we have $[\hat{\theta}_\rho(x), \hat{\phi}_\sigma(x')] = [\hat{\theta}_\rho(x), \hat{\theta}_\sigma(x')] = [\hat{\phi}_\rho(x), \hat{\phi}_\sigma(x')] = 0$.

Then the charge density/current operators are

$$\begin{aligned}
\hat{\rho}_{+}(x) &= \frac{1}{\sqrt{2}}(\hat{\rho}_R + \hat{\rho}_L) \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(\hat{\rho}_{R\uparrow} + \hat{\rho}_{R\downarrow}) + \frac{1}{\sqrt{2}}(\hat{\rho}_{L\uparrow} + \hat{\rho}_{L\downarrow})\right) \\
&= \frac{1}{2}(\hat{\rho}_{R\uparrow} + \hat{\rho}_{R\downarrow} + \hat{\rho}_{L\uparrow} + \hat{\rho}_{L\downarrow}) \\
&\sim \frac{1}{2}\frac{1}{2\pi}\partial_x\left(-\hat{\Phi}_{R\uparrow} - \hat{\Phi}_{R\downarrow} + \hat{\Phi}_{L\uparrow} + \hat{\Phi}_{L\downarrow}\right) \\
&= \frac{1}{4\pi}\partial_x\left(-\sqrt{2}\hat{\Phi}_{R\rho} + \sqrt{2}\hat{\Phi}_{L\rho}\right) \\
&= -\frac{\sqrt{2}}{4\pi}\partial_x\left(2\sqrt{2}\hat{\theta}_\rho\right) \\
&= -\frac{1}{\pi}\partial_x\hat{\theta}_\rho(x), \qquad (168) \\
\hat{\rho}_{-}(x) &= \frac{1}{4\pi}\left(-\sqrt{2}\hat{\Phi}_{R\rho} - \sqrt{2}\hat{\Phi}_{L\rho}\right) \\
&= -\frac{1}{\pi}\partial_x\hat{\phi}_\rho(x), \qquad (169)
\end{aligned}$$

and the spin density/current operators are

$$\begin{aligned}
\hat{\sigma}_{+}(x) &= \frac{1}{\sqrt{2}}(\hat{\sigma}_R + \hat{\sigma}_L) \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(\hat{\rho}_{R\uparrow} - \hat{\rho}_{R\downarrow}) + \frac{1}{\sqrt{2}}(\hat{\rho}_{L\uparrow} - \hat{\rho}_{L\downarrow})\right) \\
&= \frac{1}{2}\left(\hat{\rho}_{R\uparrow} - \hat{\rho}_{R\downarrow} + \hat{\rho}_{L\uparrow} - \hat{\rho}_{L\downarrow}\right) \\
&\sim \frac{1}{2}\frac{1}{2\pi}\partial_x\left(-\hat{\Phi}_{R\uparrow} + \hat{\Phi}_{R\downarrow} + P\hat{h}i_{L\uparrow} - \hat{\Phi}_{L\downarrow}\right) \\
&= \frac{1}{4\pi}\partial_x\left(-\sqrt{2}\hat{\Phi}_{R\sigma} + \sqrt{2}\hat{\Phi}_{L\sigma}\right) \\
&= -\frac{\sqrt{2}}{4\pi}\partial_x\left(2\sqrt{2}\hat{\theta}_\sigma\right) \\
&= -\frac{1}{\pi}\partial_x\hat{\theta}_\sigma(x), \qquad (170) \\
\hat{\sigma}_{-}(x) &= \frac{1}{4\pi}\left(-\sqrt{2}\hat{\Phi}_{R\sigma} - \sqrt{2}\hat{\Phi}_{L\sigma}\right) \\
&= -\frac{1}{\pi}\partial_x\hat{\phi}_\sigma(x). \qquad (171)
\end{aligned}$$

Therefore, the final form of the phase Hamiltonian for charge and spin is

$$\begin{aligned}
\hat{H}_{\rho/\sigma} &= \hbar v_{\rho/\sigma}\int \frac{dx}{\pi} \ast\!\!\ast \left[\frac{1}{K_{\rho/\sigma}}\left(\partial_x\hat{\theta}_{\rho/\sigma}(x)\right)^2\right. \\
&\qquad \left. + K_{\rho/\sigma}\left(\partial_x\hat{\phi}_{\rho/\sigma}(x)\right)^2\right] \ast\!\!\ast\,. \qquad (172)
\end{aligned}$$

We express the phase operator $\hat{\Phi}_{\nu s}$ with $\hat{\theta}_{\rho/\sigma}$ and $\hat{\phi}_{\rho/\sigma}$. Since

$$\begin{aligned}
\hat{\Phi}_{R\rho/\sigma} &= \sqrt{2}(\hat{\phi}_{\rho/\sigma} + \hat{\theta}_{\rho/\sigma}), \qquad (173) \\
\hat{\Phi}_{L\rho/\sigma} &= \sqrt{2}(\hat{\phi}_{\rho/\sigma} - \hat{\theta}_{\rho/\sigma}), \qquad (174)
\end{aligned}$$

we have

$$\begin{aligned}
\hat{\Phi}_{L\uparrow} &= \frac{1}{\sqrt{2}}\left(\hat{\Phi}_{L\rho} + \hat{\Phi}_{L\sigma}\right) \\
&= \hat{\phi}_\rho - \hat{\theta}_\rho + \hat{\phi}_\sigma - \hat{\theta}_\sigma = \hat{\phi}_\uparrow - \hat{\theta}_\uparrow, \qquad (175) \\
\hat{\Phi}_{L\downarrow} &= \frac{1}{\sqrt{2}}\left(\hat{\Phi}_{L\rho} - \hat{\Phi}_{L\sigma}\right) \\
&= \hat{\phi}_\rho - \hat{\theta}_\rho - \hat{\phi}_\sigma + \hat{\theta}_\sigma = \hat{\phi}_\downarrow - \hat{\theta}_\downarrow, \qquad (176) \\
\hat{\Phi}_{R\uparrow} &= \frac{1}{\sqrt{2}}\left(\hat{\Phi}_{R\rho} + \hat{\Phi}_{R\sigma}\right) \\
&= \hat{\phi}_\rho + \hat{\theta}_\rho + \hat{\phi}_\sigma + \hat{\theta}_\sigma = \hat{\phi}_\uparrow + \hat{\theta}_\uparrow, \qquad (177) \\
\hat{\Phi}_{R\downarrow} &= \frac{1}{\sqrt{2}}\left(\hat{\Phi}_{R\rho} - \hat{\Phi}_{R\sigma}\right) \\
&= \hat{\phi}_\rho + \hat{\theta}_\rho - \hat{\phi}_\sigma - \hat{\theta}_\sigma = \hat{\phi}_\downarrow + \hat{\theta}_\downarrow, \qquad (178)
\end{aligned}$$

where we have introduced

$$\begin{aligned}
\hat{\phi}_s &\equiv \hat{\phi}_\rho + s\hat{\phi}_\sigma, \qquad (179) \\
\hat{\theta}_s &\equiv \hat{\theta}_\rho + s\hat{\theta}_\sigma, \qquad (180)
\end{aligned}$$

where $s = \pm$ corresponds $\uparrow / \downarrow$. Although we do not need them, their commutation relations are

$$\begin{aligned}
[\hat{\theta}_s(x), \hat{\phi}_{s'}(x')] &= [\hat{\theta}_\rho(x) + s\hat{\theta}_\sigma(x), \hat{\phi}_\rho(x') + s'\hat{\phi}_\sigma(x')] \\
&= [\hat{\theta}_\rho(x), \hat{\phi}_\rho(x')] + ss'[\hat{\theta}_\sigma(x), \hat{\phi}_\sigma(x')] \\
&= \frac{i\pi}{4}(1 + ss')\mathrm{Sgn}(x - x') \\
&= \frac{i\pi}{2}\delta_{ss'}\mathrm{Sgn}(x - x'), \qquad (181)
\end{aligned}$$

and $[\hat{\theta}_s(x), \hat{\theta}_{s'}(x')] = [\hat{\phi}_s(x), \hat{\phi}_{s'}(x')] = 0$. Corresponding Fermionic field operator is given by

$$\begin{aligned}
\hat{\Psi}(x) &= \frac{1}{\sqrt{2\pi a}}\sum_{\nu s} e^{\mp i k_F x}\hat{F}_{\nu s}e^{-i\hat{\Phi}_{\nu s}(x)} \\
&= \frac{1}{\sqrt{2\pi a}}\sum_{\nu s} \hat{F}_{\nu s}e^{\mp i(k_F x - \hat{\theta}_s(x)) - i\hat{\phi}_s(x)}. \qquad (182)
\end{aligned}$$

VII. CORRELATION FUNCTIONS

VIII. CONCLUSIONS

Acknowledgments

I thank useful discussions with T. Fujisawa, M. Hashisaka and S. Ozawa.

Appendix A: Operator identities

In this Appendix, we introduce various operator identities, where $\hat{A}$ and $\hat{B}$ are general operators and c is a c-number. $f(\hat{A})$ is an operator-valued function.

Th. I If the commutator is c-number, $[\hat{A},\hat{B}]=c$,

$$[\hat{A},e^{\hat{B}}] \;=\; ce^{\hat{B}}. \tag{A1}$$

Proof: First we prove following relation:

$$e^{-\hat{B}}\hat{A}e^{\hat{B}}=\sum_{n=0}^{\infty}\left[\hat{A},\hat{B}\right]_n, \tag{A2}$$

where the symbol $\left[\hat{A},\hat{B}\right]_n$ means that for $n\geq 0$,

$$\left[\hat{A},\hat{B}\right]_0=\hat{A}, \tag{A3}$$

$$\left[\hat{A},\hat{B}\right]_{n+1}=\left[\left[\hat{A},\hat{B}\right]_n,\hat{B}\right]. \tag{A4}$$

Then we define an operator function with a parameter s,

$$\hat{A}(s)\equiv e^{-s\hat{B}}\hat{A}e^{s\hat{B}}. \tag{A5}$$

By formally differentiate this with s, we have

$$\begin{aligned}\frac{d\hat{A}(s)}{ds}&=e^{-s\hat{B}}(-\hat{B})\hat{A}e^{s\hat{B}}+e^{-s\hat{B}}\hat{A}\hat{B}e^{s\hat{B}}\\&=e^{-s\hat{B}}\left[\hat{A},\hat{B}\right]e^{s\hat{B}}.\end{aligned} \tag{A6}$$

We can see for $n\geq 1$,

$$\frac{d^n\hat{A}(s)}{ds^n}=e^{-s\hat{B}}\left[\hat{A},\hat{B}\right]_n e^{s\hat{B}}, \tag{A7}$$

since

$$\begin{aligned}\frac{d^{n+1}\hat{A}(s)}{ds^{n+1}}&=e^{-s\hat{B}}\left[\left[\hat{A},\hat{B}\right]_n,\hat{B}\right]e^{s\hat{B}}\\&=e^{-s\hat{B}}\left[\hat{A},\hat{B}\right]_{n+1}e^{s\hat{B}}.\end{aligned} \tag{A8}$$

Then, we Taylor expand $\hat{A}(s)$ around $s=0$,

$$\begin{aligned}\hat{A}(s)&=\sum_{n=0}^{\infty}\frac{s^n}{n!}\left\{\left.\frac{d^n\hat{A}(s)}{ds^n}\right|_{s=0}\right\}\\&=\sum_{n=0}^{\infty}\frac{s^n}{n!}\left[\hat{A},\hat{B}\right]_n.\end{aligned} \tag{A9}$$

Then we set $s=1$ and we have

$$e^{-\hat{B}}\hat{A}e^{\hat{B}}=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\hat{A},\hat{B}\right]_n. \tag{A10}$$

When $\left[\hat{A},\hat{B}\right]=c$ (c-number), $\left[\hat{A},\hat{B}\right]_{n\geq 2}=0$ and

$$e^{-\hat{B}}\hat{A}e^{\hat{B}}=\hat{A}+c. \tag{A11}$$

By multiplying $e^{\hat{B}}$ from the left, we have

$$\left[\hat{A},e^{\hat{B}}\right]=ce^{\hat{B}}.\ \text{(Q.E.D.)} \tag{A12}$$

Th. II If the commutator satisfies $[\hat{A},\hat{B}]=c\hat{B}$,

$$f(\hat{A})\hat{B} \;=\; \hat{B}f(\hat{A}+c). \tag{A13}$$

Proof: From definition,

$$\hat{A}\hat{B}-\hat{B}\hat{A}=c\hat{B}, \tag{A14}$$

and hence

$$\hat{A}\hat{B}=\hat{B}(\hat{A}+c). \tag{A15}$$

By applying $\hat{A}$ from the left, we have

$$\begin{aligned}\hat{A}^2\hat{B}&=\hat{A}\hat{B}(\hat{A}+c)\\&=\hat{B}(\hat{A}+c)^2.\end{aligned} \tag{A16}$$

In general, we can prove that for $n\geq 1$,

$$\hat{A}^n\hat{B}=\hat{B}(\hat{A}+c)^n. \tag{A17}$$

Let us Taylor expand the operator-valued function $f(\hat{A})$ around zero,

$$f(\hat{A})=\sum_{n=0}^{\infty}\frac{1}{n!}\left.\frac{d^nf(x)}{dx^n}\right|_{x=0}\hat{A}^n. \tag{A18}$$

Using Eq. (A17),

$$\begin{aligned}f(\hat{A})\hat{B}&=\sum_{n=0}^{\infty}\frac{1}{n!}\left.\frac{d^nf(x)}{dx^n}\right|_{x=0}\hat{A}^n\hat{B}\\&=\hat{B}\sum_{n=0}^{\infty}\frac{1}{n!}\left.\frac{d^nf(x)}{dx^n}\right|_{x=0}\left(\hat{A}+c\right)^n\\&=\hat{B}f(\hat{A}+c).\ \text{(Q.E.D.)}\end{aligned} \tag{A19}$$

Th. III If the commutator is c-number, $[\hat{A},\hat{B}]=c$,

$$e^{-\hat{B}}f(\hat{A})e^{\hat{B}} \;=\; f(\hat{A}+c). \tag{A20}$$

Proof: From Th. I,

$$e^{-\hat{B}}\hat{A}e^{\hat{B}}=\hat{A}+c, \tag{A21}$$

and multiply each side with itself, we have

$$e^{-\hat{B}}\hat{A}^2e^{\hat{B}}=\left(\hat{A}+c\right)^2. \tag{A22}$$

Repeating this procedure, we have for $n\geq 0$,

$$e^{-\hat{B}}\hat{A}^ne^{\hat{B}}=\left(\hat{A}+c\right)^n. \tag{A23}$$

Using the Tylor expansion of operator-valued function $f(\hat{A})$,

$$\begin{aligned}e^{-\hat{B}}f(\hat{A})e^{\hat{B}}&=e^{-\hat{B}}\sum_{n=0}^{\infty}\frac{1}{n!}\left.\frac{d^nf(x)}{dx^n}\right|_{x=0}\hat{A}^ne^{\hat{B}}\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\left.\frac{d^nf(x)}{dx^n}\right|_{x=0}\left(\hat{A}+c\right)^n\\&=f(\hat{A}+c).\ \text{(Q.E.D.)}\end{aligned} \tag{A24}$$

Th. IV If the commutator is c-number, $[\hat{A}, \hat{B}] = c$, (Baker-Hausdorff-Campbel)

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}} e^{\frac{c}{2}}. \tag{A25}$$

Proof: Set $\hat{T}(s) \equiv e^{s\hat{A}} e^{s\hat{B}}$ and differentiate with s,

$$\frac{d\hat{T}(s)}{ds} = e^{s\hat{A}} \hat{A} e^{s\hat{B}} + e^{s\hat{A}} e^{s\hat{B}} \hat{B}. \tag{A26}$$

Using Th. I with $\left[\hat{A}, e^{s\hat{B}}\right] = sce^{s\hat{B}}$,

$$\hat{A} e^{s\hat{B}} = e^{s\hat{B}} \hat{A} + e^{s\hat{B}} sc. \tag{A27}$$

Therefore,

$$\frac{d\hat{T}(s)}{ds} = e^{s\hat{A}} e^{s\hat{B}} \left(\hat{A} + \hat{B} + sc\right). \tag{A28}$$

It is obvious that $\hat{T}(0) = \hat{1}$.

Alternatively, if we define $\tilde{T}(s) \equiv e^{s(\hat{A}+\hat{B})} e^{\frac{s^2 c}{2}}$ and differentiate with s,

$$\begin{aligned} \frac{d\tilde{T}(s)}{ds} &= e^{s(\hat{A}+\hat{B})} (\hat{A}+\hat{B}) e^{\frac{s^2 c}{2}} + e^{s(\hat{A}+\hat{B})} sce^{\frac{s^2 c}{2}} \\ &= e^{s(\hat{A}+\hat{B})} e^{\frac{s^2 c}{2}} \left(\hat{A} + \hat{B} + sc\right), \end{aligned} \tag{A29}$$

with $\tilde{T}(0) = \hat{1}$. Therefore, we conclude $\tilde{T}(s) = \hat{T}(s)$. Finally by putting $s = 1$, we have the given relation. (Q.E.D.)

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[3] There does not appear g_3 term, which we disregard in this note since this term represents Umklapp scatterings occurring in the system with periodic potential.