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5 Fluctuation of current

This is the lecture note on May 20, 2024 focusing on the statistics of the current in the large bias condition.

5.1 Poisson process

In a vacuum tube (and in some of solid-state devices) we get a non-steady electrical current, and it is generated by individual electrons, which are emitted from the cathode and are accelerated across a distance and deposit their charge one at a time on the anode. The electric current arising from such a process can be written

$$I(t) = \sum_{t_k} F(t - t_k), \tag{1}$$

where $F(t - t_k)$ represents the contribution to the current of an electron which arrives at time t_k. Each electron is therefore assumed to give rise to the same shaped pulse, but with an appropriate delay, as in Fig. 1.

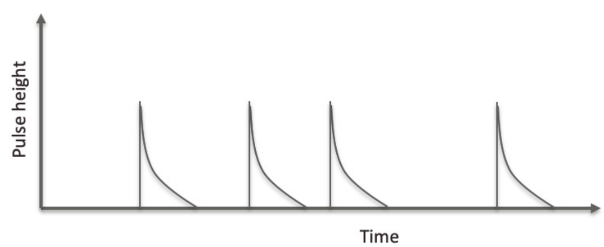

Figure 1: Illustration of shot noise: identical electron pulses arrive at random times

A statistical aspect arises immediately we consider what kind of choice must be made for t_k. The simplest choice is that each electron arrives *independently* of the previous one — that is, the times t_k are randomly distributed with a certain average number per unit time in the range $(-\infty, \infty)$, or whatever time is under consideration.[1] We shall find that there is a close connection between shot noise and processes described by birth-death master equations. For, if we consider n, the number of electrons which have arrived up to a time t, to be a statistical quantity described by a probability $P(n, t)$, with $\sum_{n=0}^{\infty} P(n, t) = 1$ for all t. The assumption that the electrons arrive independently is clearly the Markov assumption. Then, assuming the probability that an electron will arrive in the time interval between t and $t + \Delta t$

$$\mathrm{Prob}(n \to n+1, \text{in time } [t, t+\Delta t]), \tag{2}$$

is completely independent of t and n, its only dependence can be on Δt. By choosing an appropriate positive constant λ, we may write

$$\mathrm{Prob}(n \to n+1, \text{in time } \Delta t) = \lambda \Delta t, \tag{3}$$

so that for $n \geq 1$

$$P(n, t+\Delta t) = P(n,t)(1-\lambda\Delta t) + P(n-1,t)\lambda\Delta t, \tag{4}$$

and taking the limit $\Delta t \to 0$

$$\frac{\partial P(n,t)}{\partial t} = \lambda\left[P(n-1,t) - P(n,t)\right], \tag{5}$$

which is a pure birth process. For $n = 0$, the first term is missing. We write the *moment generating function* for $P(n,t)$,

$$G(s,t) = \sum_{n=0}^{\infty} s^n P(n,t). \tag{6}$$

This function is used to derive the moment of n, for example,

$$\left.\left\{\frac{d}{ds}G(s,t)\right\}\right|_{s=1} = \left.\sum_{n=1}^{\infty} n s^{n-1} P(n,t)\right|_{s=1} = \langle \hat{n} \rangle, \tag{7}$$

$$\begin{aligned}\left.\left\{\frac{d^2}{ds^2}G(s,t)\right\}\right|_{s=1} &= \left.\sum_{n=2}^{\infty} n(n-1) s^{n-2} P(n,t)\right|_{s=1} \\ &= \left\{\sum_{n=0}^{\infty} n^2 P(n,t) - P(1,t)\right\} - \left\{\sum_{n=0}^{\infty} n P(n,t) - P(1,t)\right\} = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle. \end{aligned} \tag{8}$$

By requiring at time $t = 0$ that no electrons had arrived, it is clear that $P(0,0)$ is 1 and $P(n,0)$ is zero for all $n \geq 1$, so that $G(s,0) = 1$. We find

$$\begin{aligned}\frac{\partial G(s,t)}{\partial t} &= \sum_{n=0}^{\infty} s^n \frac{\partial P(n,t)}{\partial t} = \lambda \sum_{n=1}^{\infty} s^n P(n-1,t) - \lambda \sum_{n=0}^{\infty} s^n P(n,t) \\ &= \lambda \sum_{n=0}^{\infty} s^{n+1} P(n,t) - \lambda \sum_{n=0}^{\infty} s^n P(n,t) \\ &= \lambda(s-1)G(s,t), \end{aligned} \tag{9}$$

so with using the initial condition that

$$G(s,t) = e^{\lambda(s-1)t}. \tag{10}$$

Expanding the solution Eq. (10) in powers of s,

$$G(s,t) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda s t)^n, \tag{11}$$

we find

$$P(n,t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \tag{12}$$

which is known as a *Poisson distribution*.

Using the moment generating function, we obtain several moments,

$$\langle \hat{n} \rangle = \left.\left\{\frac{d}{ds}G(s,t)\right\}\right|_{s=1} = \lambda t, \tag{13}$$

$$\langle \hat{n}^2 \rangle = \left.\left\{\frac{d^2}{ds^2}G(s,t)\right\}\right|_{s=1} + \langle \hat{n} \rangle = (\lambda t)^2 + \lambda t. \tag{14}$$

Hence, the variance of n is

$$\sigma_n^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 = \lambda t. \tag{15}$$

5.2 Fourier Analysis

The definition of the Fourier transform of a time-dependent random variable $x(t)$ is defined by

$$X(i\omega) = \int_{-\infty}^{\infty} dt x(t) e^{-i\omega t}, \tag{16}$$

and its inverse transformation is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega X(i\omega) e^{i\omega t}, \tag{17}$$

which is proved with the relation

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t-t')} = \delta(t-t'), \tag{18}$$

where $\delta(t)$ is the Dirac's delta function. Since the random variable $x(t)$ is a real number, we have the relation $X(i\omega) = X^*(-i\omega)$.

For any practical noise measurement, a measurement time interval T is finite. Even $x(t)$ is not a statistically-stationary process, the Fourier transform exists for a following gated function $x_T(t) = x(t)$ for $|t| \le T/2$ and $x_T(t) = 0$ otherwise. Therefore,[1]

$$\begin{aligned}
\int_{-\infty}^{\infty} dt x_T(t+\tau) x_T^*(t) &= \int_{-\infty}^{\infty} dt x_T(t+\tau) \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega X_T^*(i\omega) e^{-i\omega t} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega X_T^*(i\omega) \int_{-\infty}^{\infty} dt x_T(t+\tau) e^{-i\omega t} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \left| X_T(i\omega) \right|^2 e^{i\omega\tau}.
\end{aligned} \tag{19}$$

When $\tau = 0$, Eq.(19) reduces to

$$\int_{-\infty}^{\infty} dt \left| x_T(t) \right|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \left| X_T(i\omega) \right|^2, \tag{20}$$

which is a linearly increasing function of T for the statistically-stationary process. Then the average power of $x_T(t)$, statistical average of the left hand side of Eq. (20) divided by T, should have a limiting value independent of T. By first taking the ensemble average, we can exchange the order of $\lim_{T\to 0}$ and $\int_0^\infty d\omega$,

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} dt \left\langle \left| x_T(t) \right|^2 \right\rangle = \lim_{T\to\infty} \frac{1}{2\pi} \int_0^{\infty} d\omega \frac{2}{T} \left\langle \left| X_T(i\omega) \right|^2 \right\rangle \equiv \frac{1}{2\pi} \int_0^{\infty} d\omega S_x(\omega), \tag{21}$$

where we defined the *power spectral density*

$$S_x(\omega) = \lim_{T\to\infty} \frac{2 \left\langle \left| X_T(i\omega) \right|^2 \right\rangle}{T}. \tag{22}$$

For $\tau \neq 0$,

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} dt \left\langle x_T(t+\tau) x_T(t) \right\rangle \equiv \phi_x(\tau) = \frac{1}{2\pi} \int_0^{\infty} d\omega S_x(\omega) \cos\omega\tau, \tag{23}$$

where $\phi_x(\tau)$ is the ensemble averaged *autocorrelation function*. The inverse transformation is

$$4 \int_0^{\infty} d\tau \phi_x(\tau) \cos\omega\tau = \frac{2}{\pi} \int_0^{\infty} d\omega' S_x(\omega') \int_0^{\infty} d\tau \cos(\omega\tau) \cos(\omega'\tau). \tag{24}$$

[1] Since $x_T(t)$ is real, complex conjugating does nothing.

Then τ integral is

$$\begin{aligned}\int_0^\infty d\tau \cos(\omega\tau)\cos(\omega'\tau) &= \int_0^\infty d\tau \frac{1}{2}\left\{\cos(\omega-\omega')\tau + \cos(\omega+\omega')\tau\right\} \\ &= \frac{1}{4}\int_{-\infty}^{\infty} d\tau \left\{e^{i(\omega-\omega')\tau} + e^{i(\omega+\omega')\tau}\right\} \\ &= \frac{\pi}{2}\left\{\delta(\omega-\omega') + \delta(\omega+\omega')\right\}. \end{aligned} \tag{25}$$

Therefore,

$$4\int_0^\infty d\tau \phi_x(\tau)\cos\omega\tau = S_x(\omega), \tag{26}$$

which is the *Wiener-Khintchine theorem* indicating that $\phi_x(\tau)$ and $S_x(\omega)$ are the Fourier transform pairs.

Simple example of statistically-stationary noise $x(t)$, with an exponentially decaying autocorrelation function, is

$$\phi_x(\tau) = \phi_x(0)\exp\left(-\frac{|\tau|}{\tau_1}\right), \tag{27}$$

where $\phi_x(0) = \left\langle x^2 \right\rangle$ and τ_1 is the relaxation time. Then the power spectral density is

$$S_x(\omega) = \frac{4\phi_x(0)\tau_1}{1+\omega^2\tau_1^2}, \tag{28}$$

showing Lorentzian form with the cut-off frequency $\omega_c = 1/\tau_1$.

5.3 Random Pulse Train

A noisy output $x(t)$ often consists of a very large number K of random and independent discrete pulses as was introduced in Sec. 5.1 where $x(t) = I(t)$ and is represented by

$$x_T(t) = \sum_{k=1}^{K} a_k f(t-t_k), \tag{29}$$

where a_k and t_k are the k-th pulse amplitude and the time of pulse emission event, which are random variables. The function $f(t)$ is the pulse-shape function, which is determined by inherent physical properties of the system. The Fourier transform is

$$X_T(i\omega) = F(i\omega)\sum_{k=1}^{K} a_k e^{-i\omega t_k}, \tag{30}$$

where $F(i\omega) = \int_{-\infty}^{\infty} dt f(t) e^{-i\omega t}$ is the Fourier transform of $f(t)$. Then the power spectral density is given by

$$\begin{aligned} S_x(\omega) &= \lim_{T\to\infty} \frac{2\left\langle |X_T(i\omega)|^2 \right\rangle}{T} \\ &= \lim_{T\to\infty} \frac{2\,|F(i\omega)|^2}{T} \sum_{k,p=1}^{K} \left\langle a_k a_p e^{-i\omega(t_k-t_p)} \right\rangle. \end{aligned} \tag{31}$$

We then suppose $\nu = \lim_{T\to\infty} \frac{K}{T}$ as the average rate of pulse emission and assume that different pulse emission events are completely independent. We then split the sum of k and p,

$$
\begin{aligned}
S_x(\omega) &= \lim_{T\to\infty} \frac{2\left|F(i\omega)\right|^2}{T} \left\{ \sum_{k=1}^{K} \left\langle a_k^2 \right\rangle + \sum_{k\neq p} \left\langle a_k \right\rangle \left\langle a_m \right\rangle \left\langle e^{-i\omega t_k} \right\rangle \left\langle e^{i\omega t_p} \right\rangle \right\} \\
&= 2\left|F(i\omega)\right|^2 \lim_{T\to\infty} \left\{ \frac{\nu}{K} \sum_{k=1}^{K} \left\langle a_k^2 \right\rangle + \frac{1}{T} \left\langle a \right\rangle^2 K^2 \left(\frac{2\sin\frac{\omega T}{2}}{\omega T} \right)^2 \right\} \\
&= 2\nu \left\langle a^2 \right\rangle \left|F(i\omega)\right|^2 + 2\left|F(0)\right|^2 2\pi\nu^2 \left\langle a \right\rangle^2 \delta(\omega) \\
&= 2\nu \left\langle a^2 \right\rangle \left|F(i\omega)\right|^2 + 4\pi \left\{ \overline{\left\langle x(t) \right\rangle} \right\}^2 \delta(\omega),
\end{aligned} \tag{32}
$$

where we noticed

$$
\begin{aligned}
\sum_{k=1}^{K} \left\langle e^{-i\omega t_k} \right\rangle &= K \left\langle e^{-i\omega t} \right\rangle = \frac{K}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt e^{-i\omega t} \\
&= \frac{2K\sin\frac{\omega T}{2}}{\omega T},
\end{aligned} \tag{33}
$$

and $\left\langle a^2 \right\rangle = \lim_{T\to\infty} \frac{1}{K} \sum_k \left\langle a_k^2 \right\rangle$ with $\left\langle a \right\rangle = \left\langle a_k \right\rangle$. Moreover, we used the relation

$$
\lim_{T\to\infty} \frac{2\sin^2(\frac{\omega T}{2})}{\omega^2 T} = \pi\delta(\omega), \tag{34}
$$

and

$$
\overline{\left\langle x(t) \right\rangle} = \frac{1}{T} \int_{-\infty}^{\infty} dt \left\langle x_T(t) \right\rangle = \nu \left\langle a \right\rangle \int_{-\infty}^{\infty} dt f(t) \tag{35}
$$

is the mean of the noisy output $x(t)$. Equation (32) is the *Carson theorem*.

5.4 Shot noise

One particular example of the random pulse train is the steady current I. The current is assumed as the random transfer (here we assumed uni-directional, for simplicity) of the individual charge (electron) $q = -e$ where e is the unit charge. The pulse amplitudes a_k are the same $= q$ and hence $\left\langle a \right\rangle = q$ and $\left\langle a^2 \right\rangle = q^2$. The pulse-shape may depend on the physical situations but we assume $f(t) = 0$ for $t < 0$ and decays rapidly for $t > 0$ with $\int_0^\infty dt f(t) = 1$.

The average of the output is the current

$$
I \equiv \overline{\left\langle x(t) \right\rangle} = \nu q. \tag{36}
$$

The power spectrum density for $\omega > 0$ is

$$
\begin{aligned}
S_x(\omega > 0) &= 2\nu q^2 \left|F(i\omega)\right|^2 \\
&= 2\left|q\right| \left|I\right| \left|F(i\omega)\right|^2 \to_{\omega\to 0} 2\left|q\right| \left|I\right|.
\end{aligned} \tag{37}
$$

This type of noise is called *shot noise*, which is significant for non-equilibrium steady state. Important property of shot noise is that it provides the information of the charge unit that contributing the steady current by inspecting so-called *Fano factor*,

$$
\lim_{\omega\to 0} \frac{S_x(\omega)}{\left|I\right|} = 2\left|q\right|. \tag{38}
$$

In normal metals, $|q| = e$. For superconductors, $|q| = 2e$ and for the fractional quantum Hall state at filling-factor $\nu = 1/3$ (the magnetic flux in unit of magnetic flux quantum per electron is three), the charge is a fraction of unit charge, $|q| = e/3$. We will see that this simple result should be modified when there are correlations between the electrons.

5.5 Conclusion

We had explained a simple model of the current as a series of random pulse trains, which can be considered as a Poisson distribution. Fourier transform of the averaged autocorrelation function is related to the power spectral density, which is Wiener-Khintchine theorem. Power spectral density of a random pulse train can be expressed as a sum of a term with zero frequency peak and a term with the Fourier transform of each pulse peak. Finally, as a simple example, shot noise of a unidirectional current made of elemental charge q, whose Fano factor gives the information of the charge.

References

[1] S. O. Rice, "Mathematical Analysis of Random Noise".