Montague's *Intensional Logic* as Comonadic Type Theory

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Outline

Context

- Lambek (1988) proposes categorical semantics for natural language
- Reyes and Zolfaghari (1991) and Awodey et al. (2014), *inter al.*, give categorical semantics of higher order modal logic
- It makes sense to interpret *Intensional Logic* in any topos equipped with a left-exact comonad. (This subsumes the semantics of Montague (1973) and Gallin (1975))
- The structure of the models can be used to inform the formulation of the type theory.

Modal *De Dicto* and *De Re*

In PTQ

"Bill believes that a Democrat is winning."

De Dicto: $Bel(b, {}^\frown\exists x.(D(x) \wedge W(x)))$

De Re: $\exists x.(D(x) \wedge Bel(b, {}^\frown W(x)))$

"The president is necessarily the head of the executive branch."

De Dicto: $\Box\phi(p)$

De Re: $(\lambda x.\Box\phi(x))p$

Problem: Since we want to represent both *de re* and *de dicto*, we cannot have $(\lambda x.\Box\phi(x))p \equiv \Box\phi(x)[p/x] \equiv \Box\phi(p)$!

Montague's Semantics for λ

- If α is a variable, then $\alpha^{A,i,j,g}$ is $g(\alpha)$.
- If $\alpha \in ME_a$ and u is a variable of type b, then $[\lambda u.\alpha]^{A,i,j,g}$ is that function h with domain $D_{b,A,I,J}$ such that whenever x is in that domain, $h(x)$ is $\alpha^{A,i,j,g'}$, where g' is the A-assignment like g except for the possible difference that $g'(u)$ is x.

Types

- There are basic types E, T.
- If A and B are types, then so is $A \to B$.
- If A is a type, then so is $\flat A$.

Term Calculus

Variables and Function Types

- $$\frac{}{x : A \vdash x : A}$$
- $$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}$$
- $$\frac{t : A \to B \qquad u : A}{tu : B}$$

Note that we will have: $\lambda x.(tx) \equiv t \qquad (\lambda x.t)x' \equiv t[x'/x]$

Term Calculus

Logic

- $\dfrac{}{\cdot \vdash \star : T}$, where $\star$ stands for $\top$ or $\bot$
- $\dfrac{\phi : T \qquad \psi : T}{\phi \star \psi : T}$, where $\star$ stands for $\wedge$, $\vee$, or $\Rightarrow$.
- $\dfrac{\phi : T}{\neg\phi : T}$
- $\dfrac{\Gamma, x : A \vdash \phi : T}{\Gamma \vdash \star x.\phi : T}$, where $\star$ stands for $\forall$ or $\exists$.
- $\dfrac{t : A \qquad u : A}{t =_A u : T}$

Term Calculus

Modality

- $$\frac{x_1 : \flat A_1, ..., x_n : \flat A_n \vdash t : B}{x_1 : \flat A_1, ..., x_n : \flat A_n \vdash \{x_1....x_n.{}^\frown t\}(x_1, ..., x_n) : \flat B} \text{(INT)}$$
- $$\frac{}{x : \flat A \vdash {}^\smile x : A} \text{(EXT)}$$
- $$\frac{x_1 : \flat A_1, ..., x_n : \flat A_n \vdash \phi : T}{x_1 : \flat A_1, ..., x_n : \flat A_n \vdash \{x_1....x_n.\Box\phi\}(x_1, ..., x_n) : T} \text{(BOX)}$$

"The President is Necessarily the Head of the Executive Branch."

De Dicto: $\cdot \vdash \{\cdot\ \Box\phi(p)\}(\cdot)$

De Re: $\cdot \vdash \{x.\Box\phi\}(p)$

Note that $(\lambda x.\{x.\Box\phi\}(x))p \equiv \{x.\Box\phi\}(x)[p/x] \equiv \{x.\Box\phi\}(p)$

Modal *de re*/*de dicto* distinction maintained by the fact that **substitution and HAT don't commute** and **substitution and BOX don't commute**. *De re* has nothing to do with λ-abstraction!!

Comonads

(Computer Science Definition)

Definition

A **comonad** on a type theory $\mathcal{T}$ with function types consists of:

1. A type operation $\flat : Typ(\mathcal{T}) \to Typ(\mathcal{T})$
2. A type function $\epsilon_A : \flat A \to A$ for each $A \in Typ(\mathcal{T})$
3. A type function $m^* : \flat A \to \flat B$ for each $A, B \in Typ(\mathcal{T})$ and $m : \flat A \to B$

Subject to the following conditions:

1. $\epsilon_A^* = Id_{\flat A}$ for each $A \in Typ(\mathcal{T})$
2. $\epsilon_B m^* = m$ for each $A, B \in Typ(\mathcal{T})$ and $m : \flat A \to B$
3. $n^* m^* = (nm^*)^*$ for each $A, B, C \in Typ(\mathcal{T})$, $m : \flat A \to B$, and $n : \flat B \to C$

$\epsilon_{(-)}$ is the **counit** of the comonad, and $(-)^*$ may be referred to as the **lift**.

The Intension Comonad

Definition

The **intension comonad** on $\mathcal{MIL}$ is given by:

1. $\flat(-)$
2. $(\lambda x : \flat A.{}^{\vee}x) : \flat A \to A$ for each $A \in Typ(\mathcal{MIL})$
3. $(\lambda x : \flat A.\{x.{}^{\wedge}m(x)\}(x)) : \flat A \to \flat B$ for each $A, B \in Typ(\mathcal{MIL})$ and $m : \flat A \to B$

Together with the following axioms:

1. $\lambda x : \flat A.\{x.{}^{\wedge\vee}(x)\}(x) \equiv \lambda x : \flat A.x$ for each $A \in Typ(\mathcal{MIL})$.
2. $\lambda x : \flat A.{}^{\vee}\{x.{}^{\wedge}m(x)\}(x) \equiv \lambda x : \flat A.m(x)$ for each $A, B \in Typ(\mathcal{MIL})$ and $m : \flat A \to B$
3. $\lambda x : \flat A.\{y.{}^{\wedge}n(y)\}(\{x.{}^{\wedge}m(x)\}(x)) \equiv$ $\lambda x : \flat A\{x.{}^{\wedge}n(\{x.{}^{\wedge}m(x)\}(x))\}(x)$ for each $A, B, C \in Typ(\mathcal{MIL})$, $m : \flat A \to B$, and $n : \flat B \to C$

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