Constructive Hyperintensional Semantics

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Introduction

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- to point out some problems with HS that arise from limitations of the formalism in which it was written, namely classical extensional HOL,
- to introduce the calculus of inductive constructions (CIC), a modern type theory closely related to HOL,
- to explain how to fix the problems with the HOL-based implementation of HS (hereafter HHS) by shifting to a CIC-based implementation (hereafter CHS)
Outline of the talk

- Overview of static and dynamic HHS, noting some problematic aspects along the way
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- Introduction to CIC as an elaboration of HOL
- Sketch of the transition from HHS to CHS
Static HHS

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- HHS is compositional: there is a straightforward interface to (linear-logical) categorial grammar (not discussed here).
- HHS is formal: it is expressed by axioms—aka meaning postulates—in (classical extensional) HOL.
HOL: The Logic Underlying MS and HHS

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- Points of notation: Montague’s $e$ and $t$ correspond to Henkin’s $ι$ and $o$ respectively; our $w$ corresponds to Montague’s and Gallin’s $s$. 
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- For HHS, we add another basic type p (propositions) for the senses of NL sentences (cf. Thomason 1980).
HHS Types

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- This is what makes HHS \textit{hyper-intensional}: two NL expressions can have the same intension but distinct senses.
- And that in turn is why HHS is immune from various foundational problems of MS (including but not limited to logical omniscience).
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- But CIC does
The Is-True-At Relation in HHS (1/2)

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- One possible extension of HHS is to identify \( p \) with \( w \rightarrow t \) and identify \( @ \) with \( \lambda p : w \rightarrow t.p \)
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- In other words, HHS is a weaker theory than MS
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$$\vdash \forall f : A \rightarrow B . \forall w : w . f@_{A \rightarrow B}w = \lambda x : A . (f \ x)@_{B}w$$
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$$\vdash \forall f : A \rightarrow B. \forall w : w. f @ A \rightarrow B w = \lambda x : A. (f x) @ B w$$

This is yet another example of metalinguistic recursion. There is nothing in the formal theory that lets us think of these as the ‘same’ function.
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$$\lambda c : c_n. \lambda x : e^{n+1}. (c \ x_1, \ldots, x_n) \text{ and } (\text{donkey } x_0) \text{ and } (\text{bray } x_0)$$
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  - That is, a list/tuple whose components are all of the same type (e) of a certain length ($n$).

- Notice that the type of the input vector depends on a term of another type, namely natural numbers.

- The type system for HOL doesn’t allow for types to be formed in this way.

- A metalinguistic set of “vector types” (on par with our set of sense types) won’t suffice here, since we’ll want many of our semantic expressions to be defined on all contexts.
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- We want the input vector to contain semantic objects that are not necessarily entities
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- By a heterogeneous vector, we mean a term $h$ of a fixed length $n$ (as before), and for each $0 \leq i < n$, the type of the $i$th element of $h$ is also fixed (but not necessarily the same as the type of any $j$th element of $h$, $0 \leq j < n, i \neq j$)
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We call the second component the topics under discussion (TUD) stack.
The TUD-stack is similar to the QUD-stack (Ginzburg, 1994; Roberts, 1996/2012) in the sense of keeping track of accepted questions in discourse.
Question-answer anaphora: the TUD-stack

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- However, such questions are not stored as sets of propositions as in the QUD-stack but rather push onto the TUD-stack a DR for which further identification is sought.
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- The DRs on the TUD-stack form a **subvector** of the input vector.
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- HHS is a semantic theory written in HOL
- Meanings of NL sentences are treated as a primitive type (p) rather than a set of worlds (w → t)
- HHS works fine for static semantics, but proves problematic when we try to move to a dynamic system
- In general, we would like to be able to define certain families of types in the object language, as well as functions which operate on/return terms whose types are members of these type families
Curry and Feys (1958) (paraphrased): if we think of the types of pure typed lambda calculus (with $\rightarrow$ as the only type constructor) as formulas of intuitionistic propositional logic (IPL), then the IPL theorems are the types which have an inhabitant (closed term of that type)
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- In Howard’s system, arithmetic formulas (equalities and formulas built up from them using implication and quantification over natural numbers) are themselves types.
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- In Howard’s system, arithmetic formulas (equalities and formulas built up from them using implication and quantification over natural numbers) are themselves types.

- And the HA theorems are (again) the inhabited types.
Key Ideas of Howard 1969

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- The type constructors can be extended from just implication to include the other intuitionistic connectives and quantifiers
- Like terms, types can have terms as parts (including free variables)
From Howard 1969 to CIC

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  - There are variables of all types (including those whose inhabitants are types)
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- There are variables of all types (including those whose inhabitants are types).
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From Howard 1969 to CIC

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  - Some types can be inductively defined in the object language
CIC Sorts

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∀ Constructs Types of Dependent Functions

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    \( (f \ 0) : 0 = (0 + 0) \), \( (f \ 5) : 5 = (5 + 0) \), etc
- We can still use the HOL \( A \rightarrow B \) notation for types \( \forall x : A. B \) such that the variable \( x \) doesn’t occur freely in \( B \)
Type Formation Rules using $\forall$

- Systems in which there are no restrictions on forming types via $\forall$ suffer from **Girard’s Paradox** (Girard, 1972), and are therefore inconsistent.
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J. Needle, C. Pollard, and M. Yasavul

Constructive Hyperintensional Semantics
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- Additionally, certain inductive types of sort Set are standard in CIC, including nat (natural numbers) and bool (booleans/truth values)
Types for CHS

⊢ e, w, p : Set
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- Terms of type bool are used, inter alia, for the truth values of propositions at worlds.
- As in some other dynamic theories, nat is used for discourse referents (In CHS, these will be positions in argument vectors of contexts)
Going Sundholmian?

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- For us, conflating Prop and $\mathsf{p}$ would be like conflating $\mathsf{t}$ and $\mathsf{p}$ in HHS, or conflating $\mathsf{t}$ and $\mathsf{w} \rightarrow \mathsf{t}$ in MS.
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- For us, conflating Prop and p would be like conflating t and p in HHS, or conflating t and w → t in MS.
- Additionally, this would seem to predict that NL reasoning disallows the use of Excluded Middle, which strikes us as empirically incorrect.
Basic Pieces of CHS (1/2)

- Constants for entities:
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  \[ \vdash \text{and}, \text{or}, \text{implies} : p \rightarrow p \rightarrow p \]
Axioms for the connectives look similar to the ones in HHS (using $\neg, \land, \lor, \Rightarrow$ for boolean negation, conjunction, disjunction, and implication):
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\[\vdash \text{imp}_\text{ax} : \forall p, q : p. \forall w : w. [(p \text{ implies } q)@w = (p@w \Rightarrow q@w)]\]
Defining the Family of Sense Types Internally (1/2)

To allow us to refer to, and define functions over, the types of (static) linguistic senses, we first add an additional type, stat:

\[ \text{stat} : \text{Set} \]
\[ \text{ent} : \text{stat} \]
\[ \text{prp} : \text{stat} \]
\[ \text{func} : \text{stat} \to \text{stat} \to \text{stat} \]

We then define two functions, Sns and Ext to map each term of type stat to its corresponding sense/extension type:

\[ \text{Sns} \text{ent} := e \]
\[ \text{Sns} \text{prp} := p \]
\[ \text{Ext} \text{ent} := e \]
\[ \text{Ext} \text{prp} := \text{bool} \]

For any terms \( a, b : \text{stat} \),

\[ \text{Sns} (\text{func} a b) := \text{Sns} a \to \text{Sns} b \]
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Defining the Family of Sense Types Internally (1/2)

To allow us to refer to, and define functions over, the types of (static) linguistic senses, we first add an additional type, \text{stat}:

\begin{align*}
\vdash \text{stat} & : \text{Set} \\
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  \[ (\text{Sns \, ent}) := e \]
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(Ext ent) := e
(Sns prp) := p  
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Defining the Family of Sense Types Internally (2/2)

Using stat, Sns, and Ext, we can define HHS’ $@_A$ as a single recursive function language-internally:
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(\text{ext\_at } \text{ent}) := \lambda x : e. \lambda w : w. x
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Using stat, Sns, and Ext, we can define HHS’ \( @_A \) as a single recursive function language-internally:

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(\text{ext}_\text{at } \text{prp}) & := \lambda p : p. \lambda w : w. p \odot w
\end{align*}
\]

For any terms \( a, b : \text{stat} \),

\[
(\text{ext}_\text{at } (\text{func } a \ b)) := \lambda f : (\text{Sns } a) \rightarrow (\text{Sns } b). \lambda w : w. \lambda x : \text{Sns } a. \text{ext}_\text{at} b (f \ x) \ w
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Propositional Quantifiers and Equality

With stat, we can also add single constants for propositional operators corresponding to the universal & existential quantifiers, as well as for (hyperintensional) equality:
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\[ \vdash p_{\text{forall}}, p_{\text{exists}} : \forall s:\text{stat}.[((Sns \ s) \to p) \to p] \]
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We axiomatize these as follows (using ∼ for Prop negation):

\[
\vdash \text{fa} \text{ax} : \forall s: \text{stat}. \forall R: (\text{Sns } s) \rightarrow p \cdot \forall w: w. [(p_{\text{forall}} s R)^{\odot w} = \text{true}] \leftrightarrow \forall x: (\text{Sns } s). [(R x)^{\odot w} = \text{true}]
\]

\[
\vdash \text{ex} \text{ax} : \forall s: \text{stat}. \forall R: (\text{Sns } s) \rightarrow p \cdot \forall w: w. [(p_{\text{exists}} s R)^{\odot w} = \text{true}] \leftrightarrow \sim (\forall x: (\text{Sns } s). [(R x)^{\odot w} = \text{false}])
\]

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\vdash \text{eq} \text{ax} : \forall s: \text{stat}. \forall x,y: (\text{Sns } S). \forall w: w. [(x \text{equals } s y)^{\odot w} = \text{true}] \leftrightarrow (x = y)
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(Homogeneous) Vectors

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- Of particular interest to us will be dependent pairs $\langle n, v \rangle$
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- Call the type of such pairs **Arity**
Heterogeneous Vectors (1/2)

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  ⊢ HetVect : Arity → Set
  ⊢ hnil : HetVect ⟨0, [ ]⟩
  ⊢ hcons : ∀⟨n, v⟩:Arity.∀s:stat.
  [Sns s → HetVect ⟨n, v⟩ → HetVect ⟨S n, s :: v⟩]
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\[ \vdash \text{hbot} : [ ] \sqsubseteq [ ] \]

\[ \vdash \text{hcon1} : \forall \langle m, u \rangle, \langle n, v \rangle : \text{Arity}.\forall h : \text{HetVect} \langle m, u \rangle.\]
\[ \forall k : \text{HetVect} \langle n, v \rangle.\]
\[ [h \sqsubseteq k \rightarrow \forall s : \text{stat}.\forall x : \text{Sns} s. [h \sqsubseteq (x :: k)]] \]
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\[ \vdash \text{hcon2} : \forall \langle m, u \rangle, \langle n, v \rangle : \text{Arity}. \forall h : \text{HetVect} \langle m, u \rangle. \forall k : \text{HetVect} \langle n, v \rangle. \]

\[ [h \sqsubseteq k \rightarrow \forall s : \text{stat}. \forall x : \text{Sns} s. [(x :: h) \sqsubseteq (x :: k)]] \]
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The proof editing mode greatly simplifies the process of defining the functions of, and proving theorems about, CHS
Conclusion

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- While HOL is the logical framework most familiar to many semanticists, it isn’t powerful enough to be suited for the task of analyzing NL expressions.
- Moving to CIC allows us to work in a framework which is both similar enough to HOL that we can port what worked there to the new system, yet powerful enough to allow us to analyze a wider range of phenomena.
References I


References II


A Quandary about Formulas (1/2)

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- But in CIC, this is no longer the case: formulas are ‘more intensional’ than in classical extensional HOL.
- In fact, they are themselves \textit{types}.
- This fact gives rise to the option of using formulas not only to express the theory, but also as the translations of NL declarative sentences.
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In terms of models: the interpretation of a formula is just its truth value.

But in CIC, this is no longer the case: formulas are ‘more intensional’ than in classical extensional HOL.

In fact, they are themselves \textit{types}.

This fact gives rise to the option of using formulas not only to express the theory, but also as the translations of NL declarative sentences.

And most researchers using CIC for NL semantics exercise that option.
This approach is a version of Sundholm’s (1986) idea of analyzing NL sentence meanings as types.
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Should CHS follow suit?
The Inductive Type nat (1/2)

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That is, a fully normalized term of type nat will be either the nullary constant 0 or the unary constructor \( S \) applied to another term \( i : \text{nat} \).
nat also comes with three elimination schemes—\texttt{nat\_rect}_i, \texttt{nat\_rec}, and \texttt{nat\_ind} for eliminating into the sorts Type\_i, Set, and Prop, respectively:

\begin{align*}
nat\_ind : \forall P : \text{nat} \to \text{Prop}.[(P \ 0) \to \forall i : \text{nat}.[P \ i \to P \ (S \ i)]] & \to \forall n : \text{nat}.[P \ n]\
\end{align*}
The Inductive Type nat (2/2)

- nat also comes with three elimination schemes—nat_rect_i, nat_rec, and nat_ind for eliminating into the sorts Type_i, Set, and Prop, respectively:
  
  \[
  \text{nat_ind} : \forall P : \text{nat} \to \text{Prop}.\left((P \ 0) \to \forall i : \text{nat}.[P \ i \to P \ (S \ i)]\right) \to \forall n : \text{nat}.[P \ n]\]

- These destructors reduce eliminate in the expected way—ie for any \( P : \text{nat} \to \text{Prop} \), \( b : (P \ 0) \), \( f : \forall i : \text{nat}.[P \ i \to P \ (S \ i)] \), \( n : \text{nat} \):
The Inductive Type nat (2/2)

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  \[ \text{nat_ind} : \forall P : \text{nat} \to \text{Prop}.[(P 0) \to \forall i : \text{nat}.[P i \to P (S i)]] \to \forall n : \text{nat}.[P n] \]

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  \[ (\text{nat_ind} P b f 0) \Rightarrow 0 \]
The Inductive Type nat (2/2)

- nat also comes with three elimination schemes—\texttt{nat\_rect}_i, \texttt{nat\_rec}, and \texttt{nat\_ind} for eliminating into the sorts Type, Set, and Prop, respectively:

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  \texttt{nat\_ind} : \forall P : \text{nat} \to \text{Prop}.[(P 0) \to \forall i : \text{nat}.[P i \to P (S i)]] \to \forall n : \text{nat}.[P n]
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- These destructors reduce eliminate in the expected way—ie for any \( P : \text{nat} \to \text{Prop}, b : (P 0), f : \forall i : \text{nat}.[P i \to P (S i)], n : \text{nat}:

  \begin{align*}
  & \text{(nat\_ind } P b f 0) \Rightarrow 0 \\
  & \text{(nat\_ind } P b f (S n)) \Rightarrow (f n (\text{nat\_ind } P b f n))
  \end{align*}